# SOME APPLICATIONS OF PROJECTIONS IN NONLINEAR CONTROL AND ESTIMATION

by

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## SUMMARY

The present thesis uses an approach which regards nonlinear systems as a pair $(z(\cdot),z_f)$ trajectory-final state, for the case of controllability or $(z_0,z(\cdot))$ initial state-trajectory, for the case of state estimation, in a space M which is the cross product $\mathfrak{F} \times Z$ or $Z \times \mathfrak{F}$ between the space of trajectories $\mathfrak{F}$ and the state space $Z$. In this setting some mappings $F:M \rightarrow M$ are constructed using projections $P$ onto specific subsets $S$ of $M$ (i.e. $P^2 = P$ and $R(P) = S$). The solution of the problems of nonlinear controllability, state estimation and state and parameter estimation are obtained via the fixed points of such F's.

Fixed points theorems have been used in [8, 9, 25, 35 and 15] to provide global controllability, state estimation and joint state and parameter estimation. Some theoretical results are presented here, which show that it is possible to eliminate some of the assumptions which restricted the systems treated in these papers and at the same time to obtain mappings with fixed points which contain all the possible solutions for the problem of nonlinear controllability, state estimation and the joint state and parameter estimation. Among the relaxations allowed now are, for example, in the problem of control, the possibility of a set of admissible controls $\mathcal{U}_{ad}$ different from the set $\mathcal{U}$ of all input controls of the system.

In order to obtain continuous projections $P$, $S$ must be closed in $M$. This will occur naturally in the case of state estimation however, for the control case, in general, it will be necessary to adjust the topologies of the spaces $\mathcal{U}$ and/or $M$ in order to achieve this. A comprehensive theory which shows that this adjustment is always possible as well as a complete procedure for obtaining the adjusted spaces $\mathcal{U}$ and $M$ are presented here.



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# CHAPTER 1.

## INTRODUCTION.

Fixed point theorems together with results pertaining to linear system theory have been used in [8, 9, 25, 35 and 15] to obtain solutions for the problem of nonlinear controllability, state estimation and the joint problem state estimation and parameter identification. Here we introduce some concepts which will allow us to treat these problems, using again the approach via fixed points, with less restrictive conditions on the system.

In the case of control we assume that a set of admissible input controls $\mathcal{U}_{ad}$ , a desired final state $z_d$ and $\varepsilon \geq 0$ are given and the problem is to determine the controls $u^* \in \mathcal{U}_{ad}$ , which drive the system from the initial state $z_0$ at $t = 0$ to a final state $z_f$ at $t = T$ such that

$$\|z_f - z_d\| \leq \varepsilon \quad .$$

In particular, if the given $\varepsilon$ is zero, the controls $u^*$ will drive the system to $z_d$ in time $T$ . For the sake of simplicity we assume that systems have the form

$$\dot{z} = Az + \mathcal{B}u + \mathcal{N}z \ , \quad z(0) = z_0 \in Z \tag{1.1}$$

where the linear operator $A$ generates a strongly continuous semigroup $S(\cdot)$ on a Banach space $Z$, the state space, $B:U \to Z$ and $Nz$ denotes the nonlinearity. System (1.1) may, for instance, be derived from a linearisation of a system described by nonlinear evolution equations (see [9, 25 and 35]) as for example

$$\dot{z}(t) = f(z,u,t) \quad , \quad z(0) = z_0 \ ,$$

where $u \in \mathcal{U}_{ad} \subset \mathcal{U}$ is the input of the system which is known and $\mathcal{U}$ is a space of functions from $[0,T]$ to a Banach space $U$. Results may also be obtained for more general types of systems and this is indicated in section 6.6.

For the problem of state estimation a simplified structure is considered

$$\dot{z} = Az + Nz \qquad z(0) = z_0 \in Z \tag{1.2}$$

$$y_e(t) = Cz(t) \tag{1.3}$$

where $C \in \mathcal{L}(Z,Y)$, the output space $Y$ is a Banach space and it is assumed that the observation $y_e$ belongs to a space $\mathcal{Y}$ of functions from $[0,T]$ to $Y$. Again, the results may be extended to more general systems. System (1.2)-(1.3) may be the linearisation of a system described by nonlinear evolution equations (see [8, 9 and 35]) as for example

$$\dot{z}(t) = f(z,u,t) \qquad z(0) = z_0$$

$$y_e(t) = h(z,u,t) \ .$$

The problem of state estimation is to construct the state $z^*(t)$, $t \in [0,T]$ of the system when the observation $y_e$ and a possible error $\varepsilon \geq 0$ are given so that the state will satisfy

$$\|y_e - Cz^*(\cdot)\|_Y \leq \varepsilon .$$

In particular, when the given $\varepsilon$ is zero, the states $z^*(t)$ will satisfy the output equation (1.3).

For the joint problem of parameter and state estimation we consider

$$\dot{z} = Az + A_1\alpha + N(z,\alpha) , \quad z(0) = z_0 \in Z \tag{1.4}$$

where $A_1 : Z_1 \to Z$, $Z_1$ is the state of parameters, assumed to be finite dimensional and the observation process is described by (1.3) again. This system may be derived from a linearisation of a system described by (see [8, 9 and 35])

$$\begin{aligned} \dot{z}(t) &= f(z,u,\alpha,t) \qquad z(0) = z_0 \\ y_e(t) &= h(z,u,\alpha,t) . \end{aligned}$$

Now the problem is to construct the state $\dot{z}(t)$, $t \in [0,T]$ and identify the parameters $\alpha \in Z_1$ for system (1.4) when the observation $y_e$ is given.

In chapter 2 we present the main results of functional analysis, semi-groups and fixed point theorems used in this thesis. Here is also introduced the spaces $M(0,T;Z)$ which allows us to view systems as a pair $(z(\cdot),z_f)$ , trajectory-final state, for the case of control or $(z_0,z(\cdot))$ initial state-trajectory in the case of state estimation.

In chapter 3 we introduce our approach to linear and nonlinear systems. Here we also present some results related to the joint problem of state and parameter estimation. Also, this problem is put in the same frame as the problem of state estimation.

Chapter 4 presents some classical results about projections (idempotent operators) along side with some new results which are used in later chapters. Among the new concepts introduced here are: active mappings, characteristic and semi-characteristic functionals and uniform projections.

In chapter 5 we develop a comprehensive theory on what we call primitive operators and matched sets. We consider linear operators $T:\mathcal{U} \to \mathcal{X}$ where $\mathcal{U}$ and $\mathcal{X}$ are inner product spaces and show that it is always possible to adjust the topologies of $\mathcal{U}$ and/or $\mathcal{X}$ such that some desired topological properties of the operator $T$ , for example,

$T$ is a bounded operator

or

$T$ has closed range

or

$T$ is completely continuous

holds. Here is also presented a procedure to achieve this.

Finally in chapters 6 and 7 we construct the mappings $F$ which are to be used in the problemsof nonlinear controllability and state estimation respectively. The solution of these problems are obtained via the fixed points of $F$ .

# CHAPTER 2.

# FUNCTIONAL ANALYSIS.

## 2.1 - FUNCTIONALS.

Let $X$ be a normed vector space over the field $\mathbb{F}$($\mathbb{R}$ or $\mathbb{C}$) and $f:X \to \mathbb{F}$ a mapping. If $f$ is continuous then $f$ is said to be a continuous functional on $X$. If $f$ is linear transformation then $f$ is said to be a linear functional on $X$ [18].

An important class of continuous functionals is given by $C(X,\mathbb{F})$, the set of all bounded continuous functions from $X$ to $\mathbb{F}$.

An important class of linear functionals is given by $X^* = \mathcal{L}(X,\mathbb{F})$, the set of all bounded linear transformations from $X$ to $\mathbb{F}$.

If $f \in C(X,\mathbb{F})$ then

$$0 \leq |f(x)| \leq M \qquad \forall x \in X$$

for some $M < \infty$.

If $f \in X^*$ then $f$ is a continuous functional but $f \notin C(X,\mathbb{F})$.

Let $X$ be a Hilbert space, then if $f \in X^*$, there exists a unique vector $y \in X$ such that

$$f(x) = \langle x,y \rangle \qquad \text{for all } x \in X .$$

This result is known as the Riesz Representation theorem [3,32,39].

2.1.1 - Example.

Here we present some types of functionals which will appear in chapters 6 and 7. Let $Z$ be a normed vector space over $\mathbb{F}$. For a given $\varepsilon \geq 0$ and $z_d \in Z$ we shall need a continuous functional $q: Z \to \mathbb{F}$ on $Z$ which has the property

$$q(x) = 0 \quad \Longleftrightarrow \quad x \in \overline{B_\varepsilon(z_d)} \tag{2.1.1}$$

where $\overline{B_\varepsilon(z_d)}$ is the closed ball of radius $\varepsilon$ and centred in $z_d$.

Some examples of functional $q$ which satisfy (2.1.1) are:

i) $$q(x) = \begin{cases} \|x - z_d\|_Z - \varepsilon & \text{if } \|x - z_d\|_Z > \varepsilon \\ 0 & \text{if } \|x - z_d\|_Z \leq \varepsilon \end{cases} \tag{2.1.2}$$

ii) $$q(x) = |\varepsilon - \|x - z_d\|_Z| - \varepsilon + \|x - z_d\|_Z \tag{2.1.3}$$

iii) $$q(x) = \begin{cases} \dfrac{m(\|x-z_d\|_Z - \varepsilon)}{\|x-z_d\|_Z + \varepsilon} & \text{if } \|x-z_d\|_Z > \varepsilon \\ 0 & \text{if } \|x_f-z_d\|_Z \leq \varepsilon \end{cases} \tag{2.1.4}$$

for some $m \neq 0$.

The functional $q$ in (2.1.4) satisfies

$$0 \leq |q(x)| < m$$

for any $m < \infty$, i.e., $q \in C(X,\mathbb{R})$.

□

## 2.2 - OPERATORS.

There are numerous texts in functional analysis. For nonlinear operators we refer to [26] and for linear operators we refer to [3,14,16,18,32,38,39].

Let $\mathcal{U}$ and $\mathcal{X}$ be two vector spaces over the field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). A function T from its domain $\mathcal{D}(T) \subset \mathcal{U}$ into $\mathcal{X}$ is called an operator from $\mathcal{U}$ to $\mathcal{X}$. The class of all operators from $\mathcal{U}$ to $\mathcal{X}$ is represented by $\mathcal{O}p(\mathcal{U},\mathcal{X})$.

$T \in \mathcal{O}p(\mathcal{U},\mathcal{X})$ is said to be a linear operator (or linear transformation) if $\mathcal{D}(T)$ is a vector subspace of $\mathcal{U}$ and

$$T(ax+by) = aTx + bTy$$

holds for all $x,y \in \mathcal{D}(T)$, $a,b \in \mathbb{F}$.

The class of all linear operators from $\mathcal{U}$ to $\mathcal{X}$ is denoted by $\mathcal{LO}p(\mathcal{U},\mathcal{X})$. If $T \in \mathcal{O}p(\mathcal{U},\mathcal{X})$ but $T \notin \mathcal{LO}p(\mathcal{U},\mathcal{X})$ then T is called a nonlinear operator from $\mathcal{U}$ to $\mathcal{X}$.

For $T \in \mathcal{O}p(\mathcal{U},\mathcal{X})$, $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are defined by

$$\mathcal{N}(T) = \{u \in \mathcal{D}(T) : Tu = 0 \in \mathcal{X}\}$$

$$\mathcal{R}(T) = \{Tu \in \mathcal{X} : u \in \mathcal{D}(T)\} .$$

If $T \in \mathcal{LO}p(\mathcal{U},\mathcal{X})$ then $\mathcal{N}(T)$, called the null space of T, and $\mathcal{R}(T)$, called the range of $T$, are subspaces of $\mathcal{U}$ and $X$ respectively. If $\mathcal{U}_{ad} \subset \mathcal{D}(T)$ and $S \subset X$ then $T(\mathcal{U}_{ad})$ and $T^{-1}(S)$ are defined by

$$T(\mathcal{U}_{ad}) = \{Tu : u \in \mathcal{U}_{ad}\}$$

and

$$T^{-1}(S) = \{u \in \mathcal{D}(T) : Tu \in S\} \quad .$$

Some algebraic operations are also defined for operators:

Let $T, T' \in \mathcal{O}p(\mathcal{U},X)$. For $a \in \mathbb{F}$, $aT \in \mathcal{O}p(\mathcal{U},X)$ is defined by $(aT)u = aTu$ with $\mathcal{D}(aT) = \mathcal{D}(T)$. Also, if $\mathcal{D}(T) \cap \mathcal{D}(T') \neq \emptyset$ (the empty set) then $(T+T') \in \mathcal{O}p(\mathcal{U},X)$ is defined by $(T+T')u = Tu + T'u$ with $\mathcal{D}(T+T') = \mathcal{D}(T) \cap \mathcal{D}(T')$. Finally, if $\mathcal{Y}$ is a vector space over $\mathbb{F}$ and $C \in \mathcal{O}p(X,\mathcal{Y})$, $CT \in \mathcal{O}p(\mathcal{U},\mathcal{Y})$ is defined by $(CT)u = C(Tu)$ and $\mathcal{D}(CT) = T^{-1}(\mathcal{D}(C))$.

The notation $\mathcal{O}p(X)$ is used in cases $\mathcal{U} = X$.

The class of operators $T \in \mathcal{O}p(\mathcal{U},X)$ such that $\mathcal{D}(T) = D \subset \mathcal{U}$ is represented by $\mathcal{F}(D,X)$. $\mathcal{F}(D,X)$ is closed under addition and scalar multiplication. In fact $\mathcal{F}(D,X)$ is a vector space over $\mathbb{F}$. If $\mathcal{L}\mathcal{F}(D,X)$ is the subclass of all $T \in \mathcal{F}(D,X)$ such that $T \in \mathcal{L}\mathcal{O}p(\mathcal{U},X)$ then $\mathcal{L}\mathcal{F}(D,X)$ is a vector subspace of $\mathcal{F}(D,X)$.

Now suppose that $(\mathcal{U},\|\cdot\|_{\mathcal{U}})$ and $(X,\|\cdot\|_X)$ are normed vector spaces. It is well known that for $D \subseteq \mathcal{U}$, $T \in \mathcal{F}(D,X)$ and $E$ a subset of $X$, if $T$ is continuous, then

$$T^{-1}(E) \text{ is open} \iff E \text{ is open} \tag{2.2.1}$$

$$T^{-1}(E) \text{ is closed} \iff E \text{ is closed} \tag{2.2.2}$$

and also, if $\mathcal{U}_{ad} \subset D$ is a compact set, then

$$T(\mathcal{U}_{ad}) \text{ is compact.} \tag{2.2.3}$$

By (2.2.2), $N(T)$ is closed whenever $T$ is continuous.

If $T \in LOp(\mathcal{U},X)$ then $T$ maps convex sets of $\mathcal{D}(T)$ into convex sets of $X$.

2.2.1 - <u>Completely continuous mappings.</u>

A mapping $T:D \subseteq \mathcal{U} \to X$ is said to be compact if $\overline{T(\mathcal{U}_{ad})}$ is a compact subset of $X$ for any bounded set $\mathcal{U}_{ad} \subset D$. $T$ is also said to be completely continuous if it is both compact and continuous.

An equivalent definition of completely continuous mappings is: $T$ is completely continuous if it maps weakly convergent sequences in $D$ into strongly convergent sequences in $X$.

Clearly if $R(T) \subset E$ and $E$ is compact then $T$ is compact. Also, if $T \in LOp(\mathcal{U},X)$ then $T$ is compact if and only if $T$ is completely continuous. If $T$ has range $R(T) \subset E$ and $E$ is a finite dimensional subspace of $X$ then $T$ is compact.

Let $q : X \to \mathbb{F}$ be any functional on $X$ and $Q : D \to X$ , $D \subseteq X$ , the mapping

$$Q(x) = q(x)\bar{x} \tag{2.2.4}$$

where $\bar{x} \in X$ is a fixed element. Clearly $Q$ is compact since $R(Q)$ is a one-dimensional space. If $q$ is a continuous functional then $Q$ is completely continuous.

2.2.2 - Some nonlinear operators.

An important class of not necessarily linear operators is given by the Lipschitz operators from $D$ to $Y$ for $X$ and $Y$ normed vector spaces over $\mathbb{F}$ and $D \subset X$ . These are the elements $N \in \mathfrak{F}(D,Y)$ for which there is a number $\ell \geq 0$ such that

$$\|Nx - Ny\|_Y \leq \ell \cdot \|x - y\|_X \quad \text{for all} \quad x,y \in D . \tag{2.2.5}$$

Another important class of operators is given by the members $N \in \mathfrak{F}(D,Y)$ which satisfy the Lipschitz-type condition

$$\|Nx - Ny\|_Y \leq k(\|x\|_X \, , \|y\|_X) \cdot \|x-y\|_X \quad \forall \, x,y \in D \tag{2.2.6}$$

where $k(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, symmetric and $k(0,0) = 0$ .

In this case we define the constants $k_N$ and $\bar{k}_N$ by

$$\bar{k}_N = \sup_{x \in D}\{k(\|x\|,0)\} \tag{2.2.7}$$

and

$$k_N = \sup_{x,y \in D}\{k(\|x\|, \|y\|)\} \quad . \tag{2.2.8}$$

Remark:

In cases where $X$ is a space of functions of the type $\mathcal{F}(D',Z)$ (e.g. $D' = [a,b]$ , $X = L^p(a,b;Z)$ or $X = C(a,b;Z)$ , where $Z$ is a normed vector space), then most nonlinearities map $Z$ into a larger space. The class of operators $N$ in (2.2.6) allows for a large number of such nonlinearities to be considered.

2.2.3 - Linear integral operators.

An important class of linear operators is given by integral operators of both Fredholm-type and Volterra-type. Let $X$ be a normed vector space of functions $x(\cdot) \in \mathcal{F}(D',Z)$ with $D' = [a,b]$ an interval of the real line and $Z$ being a normed vector space (e.g., $X = C(a,b;Z)$ or $X = L^p(a,b;Z)$) . Elements of $X$ are called abstract functions which we shall often denote by $x(\cdot)$ . Similarly, if $Y$ is another space of abstract functions from $D'$ to $Z$ we denote elements of $Y$ by $y(\cdot)$ .

Let the integral operator $T \in LOp(Y,X)$ be given by $Ty(\cdot) = x(\cdot)$

where

$$x(t) = \int_a^b K(s,t)\, y(s)ds \tag{2.2.9}$$

and $K(s,t) \in \mathcal{LO}p(\mathcal{Y},\mathcal{X})$ for each $(s,t) \in [a,b] \times [a,b]$ . $\mathcal{T}$ is an example of a Fredholm-type operator. In the particular case $K(s,t) = 0$ for $s > t$ , equation (2.2.9) becomes

$$x(t) = \int_a^t K(s,t)\, y(s)ds \ . \tag{2.2.10}$$

In this case $\mathcal{T}$ is said to be a Volterra-type operator.

Now let $a = 0$ , $b = T$ and consider the Volterra-type operators (2.2.10) with

$$K(s,t) = \mathcal{S}(t-s)$$

where $\mathcal{S}(t)$ is a strongly continuous semigroup on $Z$ (see section 2.3). In this case we shall use the notation $L(\cdot)$ for the operator $T$ and $L(t)\, y(\cdot)$ for $(L(\cdot)y(\cdot))(t)$ , that is $L(\cdot) \in \mathcal{LO}p(\mathcal{Y},\mathcal{X})$ , $L(\cdot)y(\cdot) = x(\cdot)$ and $L(t) \in \mathcal{LO}p(\mathcal{Y},Z)$ for each $t$ , $L(t)y(\cdot) = x(t)$ where

$$x(t) = \int_0^t \mathcal{S}(t-s)y(s)ds \ . \tag{2.2.11}$$

In particular, when $t = T$ , $L(T):\mathcal{Y} \to Z$ is given by $L(T)y(\cdot) = z_1$

$$z_1 = \int_0^T S(t-s)y(s)ds \quad . \tag{2.2.12}$$

This terminology will simplify the notation in later chapters.

Suppose now that $L(\cdot)$ and $L(T)$ satisfy

$$||L(\cdot)y(\cdot)||_X \leq k_1' \; ||y(\cdot)||_Y \tag{2.2.13}$$

and

$$||L(T)y(\cdot)||_Z \leq k_2' \; ||y(\cdot)||_Y \tag{2.2.14}$$

for some constants $k_1'$ and $k_2' > 0$. This implies that if $N$ satisfies (2.2.6) then $L(\cdot)N$ and $L(T)N$ satisfy the following Lipschitz-type condition

$$||L(\cdot)Nx(\cdot) - L(\cdot)Nx'(\cdot)||_X \leq c_1 \; ||x(\cdot) - x'(\cdot)||_X \tag{2.2.15}$$

and

$$||L(T)Nx(\cdot) - L(T)Nx'(\cdot)||_Z \leq c_2 \; ||x(\cdot) - x'(\cdot)||_X \tag{2.2.16}$$

for all $x(\cdot), x'(\cdot) \in D = \mathcal{D}(N) \subset X$, where $c_1 = k_N k_1'$ and $c_2 = k_N k_2'$. Moreover,

$$||L(\cdot)Nx(\cdot)||_X \leq \bar{c}_1 \; ||x(\cdot)||_X \qquad \forall \; x(\cdot) \in D \tag{2.2.17}$$

and

$$\|L(T)Nx(\cdot)\|_Z \leq \bar{c}_2 \ \|x(\cdot)\|_X \qquad \forall\, x(\cdot) \in D \tag{2.2.18}$$

where $\bar{c}_1 = \bar{k}_N k_1'$ and $\bar{c}_2 = \bar{k}_N k_2'$ .

2.2.4 - Bounded linear operators.

Let $\mathcal{U}$ and $X$ be normed vector spaces over $\mathbb{F}$ and $\mathcal{L}(\mathcal{U},X)$ be the class of all $T \in LOp(\mathcal{U},X)$ such that $\mathcal{D}(T) = \mathcal{U}$ and there is a real number $m \geq 0$ such that

$$\|Tu\|_X \leq m\,\|u\|_{\mathcal{U}} \qquad \forall\, u \in \mathcal{U} \ .$$

If $\mathcal{U} = X$ it is usual to represent $\mathcal{L}(X,X)$ by $\mathcal{L}(X)$ .

A classical result in functional analysis is that if $T \in \mathcal{L}(\mathcal{U},X)$ then $T$ is continuous and vice-versa. That is, $\mathcal{L}(\mathcal{U},X)$ is also the class of all continuous linear transformations from $\mathcal{U}$ to $X$ .

The above definition says that a bounded (or continuous) linear transformation maps bounded sets into bounded sets. Another classical result is that $\mathcal{L}(\mathcal{U},X)$ is a normed vector space over $\mathbb{F}$ with the norm given by

$$\|T\| = \sup_{u \neq 0} \frac{\|Tu\|}{\|u\|} \ .$$

If $T \in \mathcal{L}(\mathcal{U},X)$ and $C \in \mathcal{L}(X,\mathcal{Y})$ then the composition $CT \in \mathcal{L}(\mathcal{U},\mathcal{Y})$ and satisfies

$$||CT|| \le ||C|| \cdot ||T|| \quad . \tag{2.2.19}$$

Clearly if $L(\cdot) \in \mathcal{L}(\mathcal{Y},X)$ then $L(\cdot)$ satisfies (2.2.13) for any $k_1' \ge ||L(\cdot)||$ and similarly, if $L(T) \in \mathcal{L}(\mathcal{Y},Z)$ , $L(T)$ satisfies (2.2.14) for any $k_2' \ge ||L(T)||$ .

If $T$ is a linear operator but $T \notin \mathcal{L}(\mathcal{U},X)$ then $T$ is said to be an unbounded operator from $\mathcal{U}$ to $X$ .

2.2.5 - Example:

Let $\mathcal{U}$ and $Z$ be normed vector spaces, $B \in \mathcal{LO}p(\mathcal{U},\mathcal{F}(D',Z))$ , $D' = [0,T]$ , $X$ and $\mathcal{Y}$ be two normed vector spaces of abstract functions defined on $D'$ with values in $Z$ , $L(\cdot) \in \mathcal{L}(\mathcal{Y},X)$ and $L(T) \in \mathcal{L}(\mathcal{Y},Z)$ .

If

$$B \in \mathcal{L}(\mathcal{U},\mathcal{Y})$$

then, by (2.2.19),

$$||L(\cdot)Bu||_X \le \tilde{k}_1 \, ||u||_{\mathcal{U}} \tag{2.2.20}$$

and

$$||L(T)Bu||_Z \le \tilde{k}_2 \, ||u||_{\mathcal{U}} \tag{2.2.21}$$

for some $\tilde{k}_1,\tilde{k}_2 > 0$ . That is, $L(\cdot)B \in \mathcal{L}(\mathcal{U},X)$ and $L(T)B \in \mathcal{L}(\mathcal{U},Z)$ .

Observe that the above condition on $B$ may be weaker than

$$B \in \mathcal{L}(\mathcal{U},X)$$

if for example $\mathcal{Y}$ is a larger space than $X$ . That is, $B:\mathcal{U} \to X$ may be unbounded but (2.2.20) and (2.2.21) still hold. □

2.2.6 - <u>The spaces M(a,b;Z) and $M(a,b;Z)$</u> .

Here we introduce the spaces $X = M(a,b;Z)$ which will be vaguely defined as being the cross product between $\mathfrak{F}(D',Z)$, $D' = [a,b]$ and the normed vector space $Z$ . The definition is rather loose since we admit $X$ being either

$$\mathfrak{F}(D',Z) \times Z \qquad (2.2.22)$$

or

$$Z \times \mathfrak{F}(D',Z) \ . \qquad (2.2.23)$$

When dealing with these spaces we shall always use the notation $x$, $x'$, $z$, $z^*$, etc., for elements of $X$ , $x(\cdot)$, $x'(\cdot)$, $z(\cdot)$, $z^*(\cdot)$, etc., for elements of $\mathfrak{F}(D',Z)$ and $x_0$, $x_0'$, $z_0$, $z_0'$, $x_f$, $x_f'$, etc., for elements of $Z$ (e.g., $x = (x(\cdot),x_f) \in X$ or $x = (x_0,x(\cdot)) \in X$) .

If $\mathfrak{F}(D',Z) = C(a,b;Z)$ we denote by $M^0(a,b;Z)$ the Banach space $(X,\|\cdot\|_{M^0})$ where the norm $\|\cdot\|_{M^0}$ is given by

$$\|x\|_{M^0} = (\|x(\cdot)\|^2_{C(a,b;Z)} + \|x_1\|^2_Z)^{\frac{1}{2}} \qquad (2.2.24)$$

for either $x = (x(\cdot),x_1) \in C(a,b;Z) \times Z$ or $x = (x_1,x(\cdot)) \in Z \times C(a,b;Z)$ . We recall that the norm in $C(a,b;Z)$ is given by

$$\| x(\cdot) \|_{C(a,b;Z)} = \sup_{a\le t\le b} \{ \| x(t) \|_Z \} .$$

If $\mathcal{F}(D',Z) = L^p(a,b;Z)$ , $p \ge 1$ , we denote by $M^p(a,b;Z)$ the Banach space $(X, \| \cdot \|_{M^p})$ where the norm $\| \cdot \|_{M^p}$ is given by

$$\| x \|_{M^p} = (\| x(\cdot) \|_{L^p}^2 + \| x_1 \|_Z^2)^{\frac{1}{2}} \tag{2.2.25}$$

for either $x = (x(\cdot),x_1) \in L^p(a,b;Z) \times Z$ or $x = (x_1,x(\cdot)) \in Z \times L^p(a,b;Z)$. We recall that the norm in $L^p(a,b;Z)$ is defined by

$$\| x(\cdot) \|_{L^p} = \left(\int_a^b \| x(t) \|_Z^p \, dt\right)^{1/p} \qquad \text{for} \quad 1 \le p < \infty$$

and

$$\| x(\cdot) \|_{L^\infty} = \operatorname*{ess\,sup}_{a\le t\le b} \{ \| x(t) \|_Z \} .$$

For $1 \le p \le q \le \infty$ , $M^q$ is a subspace of $M^p$ . Also, $M^o$ is a subspace of $M^p$ for any $p \ge 1$ . In particular $M^o$ is a closed subspace of $M^\infty$ .

The space $M^2$ is a Hilbert space with inner product given by

$$\langle x,x' \rangle_{M^2} = \langle x(\cdot),x'(\cdot) \rangle_{L^2} + \langle x_1,x_1' \rangle_Z \tag{2.2.26}$$

for either $x = (x(\cdot),x_1)$ and $x' = (x'(\cdot),x_1')$ or $x = (x_1,x(\cdot))$ and $x' = (x_1',x'(\cdot))$ .

Now suppose $Z$ is a Hilbert space and consider the Sobolev spaces $H^s(0,T;Z)$ , $s \in \mathbb{R}$ (see [1, 22 ]). We denote $M^s(0,T;Z)$ the Hilbert spaces $H^s(0,T;Z) \times Z$ (or also $Z \times H^s(0,T;Z)$) with the norm given by

$$\| x \|_{H^s} = (\| x(\cdot) \|^2_{H^s} + \| x_1 \|^2_Z)^{\frac{1}{2}}$$

for $x = (x(\cdot),x_1) \in H^s(0,T;Z) \times Z$ (or also $x = (x_1,x(\cdot)) \in Z \times H^s(0,T;Z))$. Clearly the inner product in $M^s$ is given by

$$<x,x'>_{M^s} = <x(\cdot),x'(\cdot)>_{H^s} + <x_1,x_1'>_Z$$

for $x = (x(\cdot),x_1)$ and $x' = (x'(\cdot),x_1')$ . Also, the particular case $s = 0$ gives

$$M^0(0,T;Z) = M^2(0,T;Z) = L^2(0,T;Z) \times Z$$

2.2.7 - The operators $L$ and $S$ .

In applications of control it will be more convenient to use $M^s$ of the type (2.2.22), that is, if $x \in M^s$ then $x = (x(\cdot),x_f)$ where $x(\cdot) \in H^s(0,T;Z)$ for some $s \in \mathbb{R}$ and $x_f \in Z$ . In this case we shall use the operators $L$ and $S$ defined as follows: Let $X = H^s(0,T;Z)$ for some $s \in \mathbb{R}$ and $Y$ be a normed vector space of abstract functions from $D' = [0,T]$ to $Z$ . We define the linear operator $L:Y \to M^s(0,T;Z)$ by

$$Ly(\cdot) = (L(\cdot)y(\cdot),\ L(T)y(\cdot)) \tag{2.2.27}$$

where $L(\cdot):Y \to X$ and $L(T):Y \to Z$ were given respectively by (2.2.11) and (2.2.12) for a given strongly continuous semigroup $S(t)$. If $L(\cdot)$ and $L(T)$ satisfy (2.2.13)-(2.2.14) then $L \in \mathcal{L}(\mathcal{Y}, M^s)$. If $N: D \to \mathcal{Y}$, $D \subseteq X$, is a nonlinear operator which satisfies (2.2.2) then, $LN$ maps $D$ into $M^s$ and by (2.2.15)-(2.2.16) we have that for $x = (x(\cdot), x_f) \in M^s$ and $x' = (x'(\cdot), x'_f) \in M^s$,

$$\| LNx(\cdot) - LNx'(\cdot) \|_{M^s} \leq c \| x(\cdot) - x'(\cdot) \|_X \tag{2.2.28}$$

where $c = \sqrt{c_1^2 + c_2^2}$. Also, by (2.2.17) and (2.2.18)

$$\| LNx(\cdot) \|_{M^s} \leq \bar{c} \| x(\cdot) \|_X \tag{2.2.29}$$

where $\bar{c} = \sqrt{\bar{c}_1^2 + \bar{c}_2^2}$. If $B \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ then by (2.2.20) and (2.2.21) we have

$$\| LBu \|_{M^s} \leq \tilde{k} \| u \|_{\mathcal{U}} \tag{2.2.30}$$

where $\tilde{k} = \sqrt{\tilde{k}_1^2 + \tilde{k}_2^2}$. We also define the linear operator $S: Z \to M^s(0,T;Z)$ by

$$Sz_1 = (S(\cdot)z_1, S(T)z_1) . \tag{2.2.31}$$

In applications of state estimation we prefer to use $M^p$ of the type (2.2.23), that is, if $x \in M^p$ then $x = (x_0, x(\cdot))$ where $x_0 \in Z$

and $x(\cdot)$ is either a continuous function (when $p = 0$) or a $L^p$-function (when $p \geq 1$) from $D' = [a,b]$ to $Z$. In this case we define the operators $L$ and $S$ differently:

Let $p, X, Y, L(\cdot)$ and $S(\cdot)$ as before. We define $L: Y \to M^p(0,T;Z)$ by

$$Ly(\cdot) = (0, L(\cdot)y(\cdot)) \quad , \tag{2.2.32}$$

and $S: Z \to M^p(0,T;Z)$ by

$$Sz_0 = (z_0, S(\cdot)z_0) \ . \tag{2.2.33}$$

In this case we have that if $N: D \to Y$, $D \subseteq X$ is a nonlinear operator satisfying (2.2.6), then, by (2.2.15) and (2.2.17), $LN: D \to M^p(0,T;Z)$ satisfies (2.2.28) and (2.2.29) with $x = (x_0, x(\cdot))$, $x' = (x_0', x'(\cdot))$, $c = c_1$ and $\bar{c} = \bar{c}_1$.

2.2.8 - The extension principle.

If $T$ is a bounded linear operator from $U$ to $X$, i.e., $T \in \mathcal{L}(U,X)$, and $\mathcal{D}(T)$ is dense in $U$, then there exists a unique extension of $T$ to $\hat{T} \in \mathcal{L}(U;X)$, with $\mathcal{D}(\hat{T}) = U$ and $\|\hat{T}\| = \|T\|$. This result is known as the "extension principle". A common process to obtain $\hat{T}$ is known as "extension by continuity" [2, 3 ].

2.2.9 - Closed operators.

A linear operator $T:\mathcal{U} \to X$ is said to be closed if whenever $\{u_n\} \to u$ in $\mathcal{U}$ and $\{Tu_n\} \to x$ in $X$ one has that $u \in \mathcal{D}(T)$ and $x = Tu$ .

If $T \in \mathcal{L}(\mathcal{U},X)$ , then $T$ is closed.

The following equivalent definition is given by some authors: $T:\mathcal{U} \to X$ , linear, is closed if the graph of $T$ ,

$$\text{Graph}(T) = \{(u,Tu) : u \in \mathcal{D}(T)\} \subset \mathcal{U} \times X$$

is a closed subspace of $(\mathcal{U}\times X, \|\cdot\|_1)$ , where $\|(u,x)\|_1 = \|u\|_{\mathcal{U}} + \|x\|_X$ .

When $\mathcal{U}$ and $X$ are Banach spaces one has the closed graph theorem [38] which says that if $T:\mathcal{U} \to X$ is a closed operator with $\mathcal{D}(T) = \mathcal{U}$ , then $T \in \mathcal{L}(\mathcal{U},X)$ .

2.3 - SEMIGROUPS.

In this section we give the definition and the main properties of semigroups of operators. Any of the following references [11,17,20] contain the present material. Here $(Z,\|\cdot\|)$ is a Banach space.

2.3.1 - Semigroups and infinitesimal generators.

$\mathcal{S}(\cdot):\mathbb{R} \to \mathcal{L}(Z)$ is said to be a strongly continuous semigroup if

i) $\mathcal{S}(t+s) = \mathcal{S}(t)\mathcal{S}(s)$ , $0 \le s \le t$

ii) $\mathcal{S}(0) = I$ (identity on $Z$)

iii) $\|S(t)z_0 - z_0\| \to 0$ as $t \to 0^+$ , $\forall z_0 \in Z$ .

If $S(\cdot)$ is a strongly continuous semigroup then the set

$$\{\|S(t)\|_{\mathcal{L}(Z)} : t \in [0,T],\ T < \infty\}$$

is bounded.

Let $A:\mathcal{D}(A) \to Z$ be defined by

$$Az_0 = \lim_{t\to 0^+} \frac{S(t)z_0 - z_0}{t} \tag{2.3.1}$$

where $\mathcal{D}(A) = \{z_0 \in Z$: the limit in (2.3.1) exists$\}$. The operator $A$ is said to be the infinitesimal generator of $S(\cdot)$ . The operator $A$ is closed and densely defined in $Z$ . Actually,

$$\bigcap_n \mathcal{D}(A^n) \text{ is dense in } Z .$$

Also, if $z_0 \in \mathcal{D}(A) \Rightarrow S(t)z_0 \in \mathcal{D}(A)$ $\forall\ t \geq 0$ .

A bounded operator $A \in \mathcal{L}(Z)$ is the infinitesimal generator of the semigroup $S(\cdot)$ given by

$$S(t) = e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} . \tag{2.3.2}$$

Observe that $A$ in (2.3.1) can be equivalently expressed by

$$AS(t)z_0 = \lim_{\Delta t \to 0} \frac{S(t+\Delta t)z_0 - S(t)z_0}{\Delta t}$$

which resembles the familiar notion of derivative. We could interpret $A$ is this sense if we look to the following properties

a) $\frac{d^n}{dt^n}S(t)z_0 = A^n S(t)z_0 = S(t)A^n z_0$ , $z_0 \in \mathcal{D}(A^n)$ , $t>0$ , $n = 1,2,\ldots$

b) $S(t)z_0 - z_0 = \int_0^t S(s)Az_0 ds = \int_0^t AS(s)z_0 ds$ , $z_0 \in \mathcal{D}(A)$ .

So, if we have the abstract evolution equation

$$\dot{z}(t) = \frac{dz}{dt} = Az(t) \; , \quad z(0) = z_0 \qquad (2.3.3)$$

where $A$ is the infinitesimal generator of $S(\cdot)$ and $z_0 \in \mathcal{D}(A)$ , then the solution is given by

$$z(t) = S(t)z_0 \; .$$

Clearly, from (2.3.2), if $A \in \mathcal{L}(Z)$ then $A$ generates a semigroup. The Hille-Yosida theorem is a classical result which determines the class of unbounded operators $A$ which generate strongly continuous semigroup. Observe that $A$ must be closed and $\mathcal{D}(A)$ dense in $Z$ . For those operators $A$ the Hille-Yosida theorem says: $A$ generates a strongly continuous semigroup if and only if there exists $M$ , $\omega \in \mathbb{R}$ such that $\forall \lambda > \omega$ , $\lambda \in \mathbb{R}$, one has that

$$(\lambda I - A)^{-1} \quad \text{exists} \quad \text{and} \tag{2.3.4}$$

$$\|(\lambda I - A)^{-n}\|_{\mathcal{L}(Z)} \leq \frac{M}{(\lambda-\omega)^n} \quad , \; n = 1,2,\ldots \tag{2.3.5}$$

With the conditions of the above theorem the semigroup $S(\cdot)$ generated by $A$ satisfies

$$\|S(t)\|_{\mathcal{L}(Z)} \leq Me^{\omega t} \quad .$$

2.3.2 - Differential equations.

If instead (2.3.3) one has the inhomogeneous differential equation

$$\dot{z}(t) = Az(t) + f(t) \quad , \quad z(0) = z_0 \tag{2.3.6}$$

then a solution of (2.3.6) satisfies

$$z(t) = S(t)z_0 + \int_0^t S(t-s)f(s)ds \quad , \quad z(0) = z_0 \; . \tag{2.3.7}$$

However, not every $z(\cdot)$ which satisfies (2.3.7) will be a solution of (2.3.6). Only under certain conditions, such as for example $f \in C^1$ and $z_0 \in \mathcal{D}(A)$ , are (2.3.6) and (2.3.7) equivalent. To avoid this problem one often abandons the classical concept of solution and defines $z(\cdot)$ which satisfies (2.3.7) to be a "mild solution" of (2.3.6). It can be shown that this concept is in agreement with the concept of "weak solution" in partial differential equations. From (2.2.11) we can express the mild solution (2.3.7) by

$$z(t) = \mathcal{S}(t)z_0 + L(t)f(\cdot) \ , \quad z(0) = z_0 \tag{2.3.8}$$

## 2.4 - INNER PRODUCT SPACES.

Here it is presented a collection of results from functional analysis and theory of linear operators in inner product spaces. Any of the following texts [2,3,14,16,32,36,38,39] serves as reference. More specific reference is given in some paragraphs.

In this section $\mathcal{X}$ denotes the inner product space $(\mathcal{X},\langle,\rangle)$ over the field $\mathbb{F}(\mathbb{C}$ or $\mathbb{R})$ and $\|\cdot\|$ denotes the norm generated by $\langle,\rangle$. We shall always assume that $\mathcal{X}$ is separable.

### 2.4.1 - Orthonormal and complete orthonormal sets.

It is well known that inner product spaces have a geometry much like the familiar Euclidean geometry mainly because the concepts of orthogonality and perpendicularity. Two vector $x,y \in \mathcal{X}$ are said to be orthogonal if

$$\langle x,y\rangle = 0 \ .$$

For each set $S \subset \mathcal{X}$ the orthogonal complement $S^{\perp}$ is defined by

$$S^{\perp} = \{x \in \mathcal{X} : x \perp y \ \text{ for all } \ y \in S\} \ .$$

One can easily see that $S^{\perp}$ is a closed subspace of $X$.

A set $E$ is said to be an orthogonal set if for any pair $x,y \in E$ one has that $x \perp y$. Clearly the elements of an orthogonal set are linearly independent of each other. A classical result is that if $X$ is separable then any orthogonal set in $X$ is countable (see §5.17 of [32]). Since we shall be dealing only with separable spaces, this justifies our notation

$$E = \{\phi_n\}$$

(similar to the notation of sequences) to represent an orthogonal set. Sometimes, for clarity we denote

$$E = \{\phi_n\}_{n\in\Lambda}$$

where $\Lambda$ is a countable set of indices. The particularization for cases where $E$ has a finite number of elements is obvious.

A set $E$ is said to be orthonormal if $E$ is orthogonal and each element $\phi_n \in E$ has norm $\|\phi_n\| = 1$. An orthonormal set $E$ is said to be complete if there is no other orthonormal set $E' \neq E$ such that $E' \supset E$. It is easy to see that if $E$ is a complete orthonormal set then $E^{\perp} = \{0\}$. Other standard results are: Any separable inner product space $X$ has a complete orthonormal set $E = \{\phi_n\}$. If $E = \{\phi_n\}$ and $E' = \{\phi_n'\}$ are two complete orthonormal sets for $X$ then they have the same cardinality, that is

$$\text{card}(E) = \text{card}(E') .$$

In particular, if $X$ is finite dimension then

$$\text{card}(E) = \dim X .$$

If $E' = \{\phi_n'\}$ is an orthonormal set in $X$ then there exists another set $E''$ such that $E' \cup E''$ is a complete orthonormal set. (Clearly if $E'$ is a complete orthonormal set then $E'' = \emptyset$ .) The proof of this last assertion uses Zorn's lemma (see §9.7 of [3 ]). In section 5.4 we give a procedure for constructing $E'' = \{\phi_n''\}$ as above, when $E' = \{\phi_n'\}$ and a complete orthonormal set $E = \{\phi_n\}$ are given. We shall call $E''$ the completion of $E'$ .

If $E = \{\phi_n\}$ is a complete orthonormal set, we sometimes represent

$$[X] = \text{Span}\ \{\phi_n\} .$$

We recall that Span(E) is the set of all linear combination of a finite number of elements of $E$ (a pure algebraic concept). The above definition depends on the complete orthonormal set $\{\phi_n\}$ .

2.4.2 - Parseval's equality and Fourier series.

Let $E = \{\phi_n\}$ be a complete orthonormal set in $X$ . Then,

i) $X = \overline{\mathrm{Span}\{\phi_n\}} = \{\sum_n x_n\phi_n : x_n \in \mathbb{F}, \sum_n |x_n|^2 < \infty\}$ .

ii) The inner-product in $X$ is given by the Parseval's equality:

$$\langle x,y\rangle = \sum_n \langle x,\phi_n\rangle \overline{\langle y,\phi_n\rangle} \ . \tag{2.4.1}$$

iii) From the Parseval's equality one can express the norm of $x$ by

$$\|x\| = (\sum_n |\langle x,\phi_n\rangle|^2)^{\frac{1}{2}} \ .$$

iv) Any $x \in X$ can be expressed by the convergent series

$$x = \sum_n \langle x,\phi_n\rangle \phi_n$$

which is called Fourier series expansion of $x$ . The coefficients

$$x_n = \langle x,\phi_n\rangle$$

are called the Fourier-coefficients of $x$ .

2.4.3 - Orthonormal sets and closed subspaces.

If $S = \overline{\mathrm{Span}\{\phi_k\}}$ , $\{\phi_k\}$ an orthonormal set in $X$ , then

i) $S$ is a closed subspace of $X$ and can be expressed by

$$S = \sum_k x_k\phi_k : x_k \in \mathbf{F} \text{ and } \sum_k |x_k|^2 < \infty\} \ .$$

ii) $x \in S \Rightarrow x = \sum_k \langle x, \phi_k \rangle \phi_k$ .

iii) If $x \in X$ can be expressed by the convergent series

$$x = \sum_k x_k \phi_k$$

then $x \in S$ and $x_k = \langle x, \phi_k \rangle$ .

2.4.4 - Orthonormal sets in dense subspace.

Let $S$ be a dense subspace of $X$ and $\{\phi_n\}$ an orthonormal set in $X$ . If the Parseval's equality (2.4.1) holds for every $x \in S$ then $\{\phi_n\}$ is a complete orthonormal set. Also note that any complete orthonormal set in $S$ is a complete orthonormal set in $X$ .

2.4.5 - Complete orthonormal sets for $L^2(0,1)$.

If $X = L^2(0,1)$ over $\mathbb{R}$ then the following sets $\{\phi_n\}$ are complete orthonormal sets in $X$ :

$$\{\phi_n\} = \{\sqrt{2} \sin n\pi t\}_{n=1,2,\ldots}$$

$$\{\phi_n\} = \{\sqrt{2} \cos n\pi t\}_{n=1,2,\ldots} \cup \{1\}$$

$$\{\phi_n\} = \{1\} \cup \{\sqrt{2} \cos 2n\pi t\}_{n=1,2,\ldots} \cup \{\sqrt{2} \sin 2n\pi t\}_{n=1,2,\ldots}$$

## 2.4.6 - The Gram-Schmidt orthogonalization process.

Let $E = \{\phi_n\}$ be a countable set of linearly independent vectors in $X$ . It is always possible to obtain an orthonormal set $E' = \{\phi'_n\}$ from $E$ . The Gram-Schmidt orthogonalization process is a well known procedure which accomplishes this. $E'$ generated by this process has the property that for each $k$

$$\mathrm{Span}\{\phi_1,\ldots,\phi_k\} = \mathrm{Span}\{\phi'_1,\ldots,\phi'_k\} \quad .$$

The Gram-Schmidt process can be described by: given $E = \{\phi_n\}$ , the orthonormal set $E' = \{\phi'_n\}$ is determined by

$$\phi'_n = \frac{\tilde{\phi}_n}{\|\tilde{\phi}_n\|} \quad , \quad n = 1,2,\ldots \tag{2.4.2}$$

where

$$\tilde{\phi}_1 = \phi_1$$

$$\tilde{\phi}_k = \phi_k - \sum_{i=1}^{k-1} \langle\phi_k,\phi'_i\rangle\phi'_i \quad , \quad k > 1 \ .$$

In section 5.3 we present a generalization of this procedure to cases where $E$ is not linearly independent. (We shall call it the "generalized Gram-Schmidt process".)

2.4.7 - Parallelogram law and Pythagoras theorem.

Any two elements $x,y \in X$ satisfy

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

which is known as the parallelogram law. In the particular case $x \perp y$ one has that

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 \qquad (2.4.3)$$

which is the generalization of Pythagoras theorem in geometry.

2.4.8 - The structure of Hilbert spaces.

If $X$ is a finite dimension Hilbert space over $\mathbb{F}$ then $X$ is congruent (i.e., isometrically isomorphic) to $\mathbb{F}^n$, where $n = \dim(X)$. If $X$ is an infinite dimension Hilbert space over $\mathbb{F}$, then $X$ is congruent to $\ell_2$ over $\mathbb{F}$.

We recall that $\ell_2$ is the space of all the sequences

$$\{x_n\}_{n=1,2,\ldots} \qquad x_n \in \mathbb{F}$$

such that

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty .$$

The inner-product in $\ell_2$ is given by

$$\langle x,y\rangle = \sum_{n=1}^{\infty} x_n \, \overline{y}_n$$

for $x,y \in \ell_2$ , $x = \{x_n\}_{n=1,2,\ldots}$ and $y = \{y_n\}_{n=1,2,\ldots}$ .

2.4.9 - The weak limit.

Weakly convergent sequences in a Hilbert space $X$ have a simpler representation: If $\{x_n\} \to x$ weakly, then

$$\langle x_n,y\rangle \to \langle x,y\rangle \quad \text{for any } y \in X .$$

2.4.10 - The weak compactness property.

Every bounded sequence in a Hilbert space has a weakly convergent subsequence. (Equivalently, every bounded set in a Hilbert space is weakly compact.)

The above result is known as the weak compactness property and follows directly from the Bolzano-Weierstrass theorem (see §27 of [2 ]).

2.4.11 - Hilbert-Schmidt operators.

Let $X$ be a Hilbert space $\{\phi_n\}$ a complete orthonormal set in $X$ and $T \in \mathcal{L}(X)$ . The operator $T$ is said to be a Hilbert-Schmidt operator if

$$\sum_n \|T\phi_n\|^2 < \infty .$$

It can be shown [14] that this definition does not depend on the choice of $\{\phi_n\}$ .

If $T:X \to X$ is a Hilbert-Schmidt operator then $T$ is completely continuous [14].

The Hilbert-Schmidt norm, $|||T|||$ for a Hilbert-Schmidt operator $T$ is defined by

$$|||T||| = (\sum_n \|T\phi_n\|^2)^{\frac{1}{2}} \quad .$$

Every Hilbert-Schmidt operator $T:X \to X$ is the limit (in the topology induced by the Hilbert-Schmidt norm $|||\cdot|||$ ) of a sequence of operators $T_n:X \to X$ with $R(T_n)$ finite dimensional.

If $\|T\|$ denotes the norm of $T$ in the usual topology of $\mathcal{L}(X)$ (see §2.2.4), then

$$\|T\| \leq |||T||| \quad .$$

If $T \in \mathcal{L}(U,X)$ and $\{e_n\}$ is a complete orthonormal set in $U$ then $T$ is said to be a Hilbert-Schmidt operator if

$$\sum_n \|Te_n\|^2 < \infty \quad .$$

A Hilbert-Schmidt operator $T:U \to X$ is completely continuous [14].

## 2.5 - CONTRACTIVE-TYPE MAPPINGS AND k-SET CONTRACTIONS.

### 2.5.1 - Contractive-type mappings.

The most popular mappings of contractive-type are the contractions which appear in the Banach contraction principle (see §2.6) and several other fixed point theorems. They are defined for mappings $F:D \subset X \to X$ by: $F$ is a contraction (or a $k$-contraction) in $D$ if there exists $k \leq 1$ such that

$$\|F(x)-F(y)\| \leq k \|x-y\| \qquad \forall\, x,y \in D. \tag{2.5.1}$$

Another class of contractive-type mappings also widely studied in fixed point theorems are the non-expansive mappings (which include the contractions). $F$ is non-expansive in $D$ if it satisfies (2.5.1) with $k = 1$. Even more relaxed than non-expansive mappings are the so called locally almost non-expansive mappings. These were used by R.D. Nussbaum [33] also in the study of fixed-points. They are defined by: given any $x \in D$ and $\varepsilon > 0$ one can find a weak neighbourhood $N_x(\varepsilon)$ of $x$ in $D$ such that $F$ satisfies

$$\|F(x)-F(y)\| \leq \|x-y\| + \varepsilon \qquad \forall\, x,y \in N_x(\varepsilon) \ . \tag{2.5.2}$$

Generalised contractions are another important class of contractive-type. They were introduced by L.P. Belluce and W.A. Kirk in [6] and are defined by: $F$ is a generalized contraction in $D$ if for each $x \in D$

there exists $k(x) < 1$ such that

$$\|F(x)-F(y)\| \leq k(x)\, \|x-y\| \quad \text{for each} \quad y \in D\,. \tag{2.5.3}$$

Kirk later showed that if $F$ is continuously Fréchet-differentiable in a bounded open convex set $D$ then $F$ is a generalized contraction in $D$ if and only if $\|F'_x\| < 1$ for each $x \in D$, where $F'_x$ represents the Fréchet-derivative of $F$ at $x$.

Finally, we mention here nonlinear contractions, introduced by M.Z. Nashed and J.S.W. Wong in [30] for operators $F: X \to X$. $F$ is a nonlinear contraction on $X$ if

$$\|F(x)-F(y)\| \leq k(\|x-y\|) \qquad \forall\, x,y \in X \tag{2.5.4}$$

where now $k(\cdot)$ is a real-valued continuous function satisfying $k(r) < r$ for $r > 0$.

2.5.2 - Measure of noncompactness.

First we recall that the diameter of a set $E \subset X$, $\delta(E)$, where $X$ is a normed vector space, is defined by

$$\delta(E) = \sup_{x,y \in X} \{\|x-y\|\} \quad .$$

If $E$ is bounded then $0 \leq \delta(E) < \infty$ and $\delta(E) = 0$ if and only if

$E = \{x_0\}$ for some $x_0 \in X$. We also recall the definitions of $aE$, $a \in F$ and $E_1 + E_2$

$$aE = \{ax : x \in E\} \text{ and } E_1+E_2 = \{x+y : x \in E_1,\ y \in E_2\} .$$

Measure of noncompactness $\gamma(E)$ was defined by G. Darbo in [12] for bounded sets $E$ as

$$\gamma(E) = \inf\{d>0 : E \text{ can be covered by a finite number of sets of diameter} \le d\}.$$

Some properties, immediate from the properties of diameters are:

i) $\gamma(E) = \gamma(\overline{E})$

ii) $E \subset E' \Rightarrow \gamma(E) \le \gamma(E')$

iii) $\gamma(E) = 0 \Leftrightarrow E$ is relatively compact (i.e., $\overline{E}$ is compact).

Darbo also showed some not so immediate properties which had application in fixed point theorems:(section 2.6) If $X$ is a Banach space, then

iv) $\gamma(E_1+E_2) \le \gamma(E_1) + \gamma(E_2)$

v) $\gamma(aE) = |a|\ \gamma(E)$

vi) $\gamma(E) = \gamma(\text{convex closure of } E)$

vii) If $E_n$ is a decreasing sequence of closed, nonempty sets with $\gamma(E_n) \to 0$ , then $\cap_{n\geq 1} E_n$ is compact and nonempty.

Some of these properties were crucial for the Darbo fixed point theorem (see section 2.6).

2.5.3 - k-set contractions.

Let $F:D \to X$ , $D \subseteq X$ , $F$ continuous. Darbo [12] introduced the following concept: $F$ is a k-set contraction in $D$ if

$$\gamma(F(E)) \leq k\ \gamma(E) \tag{2.5.5}$$

for any bounded set $E \subset D$ .

Later other authors introduced other kinds of measure with similar properties also in the study of fixed point theorems.

## 2.6 - FIXED POINT THEOREMS.

One important tool in the study of the existence of a solution of a nonlinear operator equation is provided by the fixed point theorems. A classical result in this area is the Banach contraction principle presented by Banach in 1922 in his thesis [ 5 ] where he deals with complete normed spaces. It says that for some Banach space $X$ of functions if $D$ is the closure of an open bounded convex set in $X$ and $F:D \to X$ is a contraction mapping $D$ into itself then there exists a unique solution $x^*$ of

$$x = F(x) \tag{2.6.1}$$

in $D$ , called a fixed point of $F$ . This principle also provides an iterative procedure for reaching the fixed point from any $x_0 \in D$ : If one takes the sequence $\{x_n\}$ defined by

$$x_n = F(x_{n-1}) = F^n(x_0) \tag{2.6.2}$$

then $\{x_n\} \to x^*$ . This procedure is usually called successive approximation.

There have been many generalizations of this principle for mappings of contractive-type such as non-expansive mappings, generalized contractions, etc. (See §2.5.1 for definitions.) For example, let $F \in \mathcal{O}p(X)$ , if $F$ is a nonlinear contraction on $X$ (see §2.5.1) , then $F$ has a unique fixed point $x^* \in X$ and all successive approximations (2.6.2) converge to $x^*$ . Not all generalizations of the Banach contraction principle however provide an iterative procedure and some do not guarantee uniqueness of the fixed point either.

Even before Banach, in 1910, the Dutch mathematician L.E.J. Brouwer had already obtained another fixed point theorem which became the first of a series of results called topological fixed point theorems.† (In general these fixed points theorems do not offer uniqueness.) Brouwer's fixed point theorem [14] says that for $X = \mathbb{R}^n$ and $D = B_1(0)$ in $\mathbb{R}^n$ , if $F$ maps $D$ into itself then $F$ has at least one fixed point.

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† They had this name because they were mainly for mappings $F$ defined on a compact set $D$ .

Later, in 1930, the Polish mathematician P.J. Schauder developed the idea of Brouwer and established another classical result. Schauder's fixed point theorem [14] says that for $X$ a Banach space, if $F$ is completely continuous mapping $D$ into itself then $F$ has at least one fixed point. This theorem enabled Schauder and other contemporaneous mathematicians to prove existence of a solution of some differential equations which could not be proved using the contraction principle. We refer to the paper [21] which is famous in this area.

In 1955, the Italian mathematician G. Darbo defined the measure of noncompactness of a bounded set (see §2.5.2) as well as k-set contractions (see §2.5.3). Darbo's fixed point theorem [12] says that for $D$ a closed, bounded convex subset of Banach space $X$, if $F$ is a k-set contraction in $D$ with $k < 1$, then $F$ has a fixed point.

After this many authors deduced a number of fixed point theorems for contractive type mappings with perturbations. These are mappings $F$ of the type

$$F = F_1 + F_2$$

where $F_1$ is a contractive type mapping (such as a contraction, a non-expansive mapping, etc) and $F_2$ is either compact or completely continuous. Theorems of this kind usually require the satisfaction of a condition of the type

$$F_1(x) + F_2(y) \in \overline{D} \qquad \forall\, x,y \in D$$

or

$$F_1(x) + F_2(y) \in \overline{B_r(0)} \quad \text{for } x \in \partial B_r(0) \text{ and } y \in \overline{B_r(0)}$$

which is similar to the condition $F(D) \subset D$ in the earlier theorems of Banach and Schauder.

One typical result obtained from Darbo's theorem is: (see [ 33 ] If $F = F_1+F_2$ maps $D$ into $D$, $F_1$ is either a contraction or a non-expansive mapping and $F_2$ is completely continuous, then $F$ has a fixed point. It is nice to note that using this result one could deduce the Banach contraction principle (up to the existence of a fixed point), when $F_2 = 0$, and Schauder fixed point theorem, when $F_1 = 0$.

Among the numerous more recent papers which developed fixed point theorems from Darbo fixed point theorem we mention R.D. Nussbaum [ 33 ], L.P. Belluce and W.A. Kirk [ 6 ] and W.V. Petryshyn [34]. One could also find a good bibliography in the references of [34].

The Russian mathematician Petryshyn, who has papers in both fixed point theorems and iterative methods, obtained results which dropped the assumption $F(D) \subset D$, substituting for the condition: there exists a $x_0 \in D$ such that if $F(x) - x_0 = k \cdot (x-x_0)$ for some $x \in \partial D$, then $k \leq 1$. An example is the following theorem (see [34]): For $X$ a uniformly convex Banach space, $D$ a bounded open convex subset of $X$ and $F:D \to X$ satisfying the above condition, if $F = F_1+F_2$, $F_1$ a locally almost non-expansive mapping and $F_2$ a completely continuous mapping, then $F$ has a fixed point. This is one of the most relaxed fixed point theorems not only because it does not require $F$ to map $D$ into $D$ but also because the condition on $F_1$ is more relaxed than non-expansivity (see §2.5.1). However, the condition on $F$ may be difficult to check which restricts the use of the theorem in practical applications.

Apart from the method of successive approximations which is provided by some fixed point theorems there are numerous other numerical methods developed to obtained fixed points [10,19].

# CHAPTER 3.

## SYSTEMS THEORY.

Here $(Z, \|\cdot\|_Z)$ is a Banach space called the state space. Elements of $Z$ are called states and we denote them by $z_0$, $z_f$, $z_1$, etc.

### 3.1 - LINEAR SYSTEMS.

The main reference for this section is [11]. We consider linear systems in the canonical form: let $\mathcal{U}$ be a space of functions from the interval $[0,T]$ to a Banach space $U$, $\mathcal{B}:U \to Z$ be a linear operator and $A$ an operator which generates a strongly continuous semigroup $S(\cdot)$ on $Z$. A linear system is usually represented in the form

$$\dot{z}(t) = Az(t) + \mathcal{B}u(t) \ , \quad z(0) = z_0 \tag{3.1.1}$$

which is called the dynamical equation of the system. The state $z_0$ is called the initial state and $u \in \mathcal{U}$ is called an input or control of the system. Sometimes only inputs $u$ in a subset $\mathcal{U}_{ad}$ of the space of input functions $\mathcal{U}$ are allowed. $\mathcal{U}_{ad}$ is called the set of admissible controls.

Let $B \in \mathcal{L}\mathcal{O}p(\mathcal{U},\mathcal{F}(D',Z))$ , $D' = [0,T]$ be defined by

$$(Bu)(t) = \mathcal{B}\ u(t) \qquad\qquad \forall\, u \in \mathcal{U} \ .$$

By (2.3.8) the dynamical equation (3.1.1) in its mild form (usually referred to as the trajectory of the system) is given by

$$z(\cdot) = S(\cdot)z_0 + L(.)Bu \quad , \quad z(0) = z_0 \quad . \tag{3.1.2}$$

Sometimes the states $z(t)$ , $t \in [0,T]$ are measured and the observation process is usually described in the following way: Let $\mathcal{Y}$ be a space of functions from $[0,T]$ to a Banach space $Y$ and $C:Z \to Y$ be a linear operator. The output of the system $y_e \in \mathcal{Y}$ is

$$y_e(t) = Cz(t) \; . \tag{3.1.3}$$

Now let $\mathcal{C} \in \mathcal{LO}p(\mathfrak{F}(D',Z),\mathcal{Y})$ , $D' = [0,T]$ be given by

$$(\mathcal{C}z(\cdot))(t) = Cz(t)$$

then the output equation (3.1.3) is equivalent to

$$y_e = \mathcal{C}z(\cdot) \quad . \tag{3.1.4}$$

3.1.1 - <u>Controllability to a set $E \subseteq Z$</u> .

Let $z_f \in Z$ be the final state of the system, i.e.,

$$z_f = z(T) = S(T)z_0 + L(T)Bu \quad . \tag{3.1.5}$$

The dynamical equation (3.1.2) added with (3.1.5) may be represented in the compact form

$$z = \mathcal{S}z_0 + LBu \; , \quad z(0) = z_0 \; , \quad z_f = z(T) \tag{3.1.6}$$

where $\mathcal{S}$ and L are given by (2.2.31) and (2.2.27) respectively and $z = (z(\cdot), z_f)$ is the pair trajectory-final state in the space of $M(0,T;Z)$ type

$$\mathfrak{F}(D',Z) \times Z \; .$$

System (3.1.6) is said to be exactly controllable (in time T) to a set $E \subseteq Z$ if for each state $z_d \in E$ there is a control $u \in \mathcal{U}_{ad}$ such that (3.1.6) is satisfied with $z_f = z_d$. Clearly, exact controllability to E is equivalent to

$$E \subset \mathcal{S}(T)z_0 + L(T)B(\mathcal{U}_{ad}) \; .$$

System (3.1.6) is said to be approximate controllable (in time T) to E if it is exactly controllable to a dense subset of E. Approximate controllability is equivalent to

$$E \subset \mathcal{S}(T)z_0 + \overline{L(T)B(\mathcal{U}_{ad})} \; .$$

In the particular case $\mathcal{U}_{ad} = \mathcal{U}$ and $E = Z$, exact controllability to Z can be checked by the following condition: there exists $\gamma > 0$ such that

$$\| (L(T)B)^* z_1 \|_{\mathcal{U}^*} \geq \gamma \| z_1 \|_{Z^*} \qquad \forall \, z_1 \in Z \, ,$$

whereas approximate controllability to Z can be checked by the condition

$$\mathcal{N}[(L(T)B)^*] = \{0\} \; .$$

Note that $(L(T)B)^*$ , the adjoint of $L(T)B$ is given by

$$((L(T)B)^* z_1)(t) = \mathcal{B}^* \mathcal{S}^* (T-t) z_1 \quad .$$

The set

$$R = \mathcal{S}(T) z_0 + L(T)B(\mathcal{U}_{ad})$$

is usually called the reachable set of the system. Setting $E = \{z_d\}$ , a system is exactly controllable to $z_d$ if and only if $z_d \in R$ .

3.1.2 - Observability.

Consider system (3.1.2) uncontrolled, that is, $u = 0$ . Then we have

$$z(\cdot) = \mathcal{S}(\cdot) z_0 \quad , \quad z(0) = z_0 \quad , \tag{3.1.7}$$

and the output equation (3.1.3) becomes

$$y_e = C\mathcal{S}(\cdot) z_0 \quad , \quad z(0) = z_0 \quad . \tag{3.1.8}$$

Note that equation (3.1.7) is the same as

$$z = \mathcal{S} z_0 \quad , \quad z(0) = z_0 \quad , \tag{3.1.9}$$

for $\mathcal{S}$ given by (2.2.33) and $z = (z_0, z(\cdot))$ is the pair initial state-trajectory in the space

$$M(0,T;Z) = Z \times \mathfrak{F}(D',Z) \ .$$

System (3.1.9) is said to be initially observable if for each $y_e \in R(CS(\cdot)) \in Y$ there is only one $z_0 \in Z$ such that (3.1.8) holds. Clearly in this case the initial state $z_0$ and the trajectory $z(\cdot) = Sz_0$ can be determined uniquely from the observation $y_e \in Y$. Initial observability is equivalent to

$$N(CS(.)) = \{0\} \ .$$

System (3.1.9) is said to be continuously initially observable if $CS(\cdot)$ has a continuous inverse $(CS(\cdot))^{-1}$ defined on $R(CS(\cdot))$. In this case small variation in the observation $y_e$ will produce only small variations in the estimated initial state $z_0$. Continuous initial observability is equivalent to the following condition: there exists $\gamma > 0$ such that

$$\|CS(\cdot)z_0\|_Y \geq \gamma\|z_0\|_Z \qquad \forall\, z_0 \in Z \ .$$

3.1.3 - Estimated States.

For a given observation $y_e \in Y$, we define $z = (z_0, z(\cdot))$ to be an estimated state if $z_0$ satisfies (3.1.9) and

$$Cz(\cdot) = CS(\cdot)z_0 = y_e$$

which is the output equation (3.1.8). Clearly if system (3.1.9) is initially observable and $z^* = (z_0^*, z^*(\cdot))$ is the estimated state for a given output $y_e \in Y$ then $z_0^*$ is the initial state of the system and

$z^*(\cdot)$ is the trajectory of the system. However, if (3.1.9) is not initially observable there may be more than one estimated state for the same output $y_e$ but only one of them will be the actual pair initial state-trajectory.

For a given $\varepsilon \geq 0$ and $y_e \in \mathcal{Y}$ we say that $z = (z_0, z(\cdot))$ is an $\varepsilon$-estimated state if $z$ satisfies (3.1.9) and

$$Cz(\cdot) = CS(\cdot)z_0 \in \overline{B_\varepsilon(y_e)} \text{ in } \mathcal{Y} .$$

So, here we allow a possible error $\varepsilon$ in the measurements. Clearly when we set $\varepsilon = 0$ the definitions of $\varepsilon$-estimated state and estimated state coincide.

## 3.2 - NONLINEAR SYSTEMS.

Let $\mathcal{U}$, $\mathcal{U}_{ad}$ and $\mathcal{Y}$ be as defined in section 3.1. A very general formulation of a nonlinear system may be given by the evolution equation

$$\dot{z}(t) = f(z,u,t) \quad , \quad z(0) = z_0 \tag{3.2.1}$$

with output equation

$$y_e(t) = h(z,u,t) \tag{3.2.2}$$

where $u \in \mathcal{U}$ is an input (or control) and $y_e \in \mathcal{Y}$ is an output (or observation).

Suppose that, for $z_0 = \overline{z}_0$ and $u = \overline{u}$, $\overline{z}(\cdot)$ is the solution of (3.2.1) on a time interval $[0,T]$. Then, setting

$$z(\cdot) = \overline{z}(\cdot) + z'(\cdot) \quad \text{and} \quad u = \overline{u} + u'$$

one could obtain a linearised system of the type

$$\dot{z}'(\cdot) = A(t)\, z'(t) + Bu'(t) + f'(z'(\cdot),u',t) \qquad z'(0) = z_0'$$

$$y_e'(t) = y_e(t) - C\overline{z}(t) = Cz'(t) + h'(z',u',t)$$

which is time-varying in the linear part and therefore the mild solution would involve a mild evolution operator. Although it is possible to apply the techniques developed in chapters 6 and 7 to systems of the above type, technical details would mask the basic idea. For the sake of simplicity we consider in our analysis the simpler systems.

$$\dot{z}(t) = Az(t) + Bu(t) + Nz(t)\ , \ z(0) = z_0 \tag{3.2.3}$$

$$y_e = Cz(\cdot) \tag{3.2.4}$$

where the operators $A,B$ and $C$ are defined as in section 3.1 and $N$ is a nonlinear operator in $Op(\mathcal{F}(D',Z))$, $D' = [0,T]$.

Clearly the mild form of (3.2.3) is given by

$$z(\cdot) = S(\cdot)z_0 + L(\cdot)Bu + L(\cdot)Nz(\cdot)\ , \qquad z(0) = z_0\ . \tag{3.2.5}$$

3.2.1 - Controllability.

Let $z_f$ denote the final state, i.e. $z(T)$. We can add (3.2.5) with the equation

$$z_f = z(T) = S(T)z_0 + L(T)Bu + L(T)Nz(\cdot) \tag{3.2.6}$$

and then write (3.2.5)-(3.2.6) in the more compact form

$$z = Sz_0 + LBu + LNz(\cdot) \quad , \quad z(0) = z_0 \tag{3.2.7}$$

where $S$ and $L$ are given by (2.2.31) and (2.2.27) respectively and $z = (z(\cdot),z_f)$, the pair trajectory-final state, lies in a space of $M(0,T;Z)$ type.

System (3.2.7) is said to be exactly controllable to a given desired state $z_d \in Z$ if there is a control $u \in \mathcal{U}_{ad}$ such that (3.2.7) is satisfied with $z_f = z_d$. Clearly $u$ drives the system from the initial state $z_0$ at $t = 0$ to the final state $z_f = z_d$ at $t = T$.

Now consider the following more general definition of controllability to $z_d$: Given $\varepsilon \geq 0$ we say that system (3.2.7) is $\varepsilon$-controllable to $z_d$ if there exists $u \in \mathcal{U}_{ad}$ such that (3.2.7) is satisfied with

$$z_f \in \overline{B_\varepsilon(z_d)} \quad \text{in} \quad Z .$$

Clearly, if we set $\varepsilon = 0$ then $\varepsilon$-controllability to $z_d$ is equivalent to exact controllability to $z_d$.

3.2.2 - Estimated states.

Consider system (3.2.5) with $u = 0$, i.e., uncontrolled. Then

$$z(\cdot) = S(\cdot)z_0 + L(\cdot)Nz(\cdot) \quad , \quad z(0) = z_0 \tag{3.2.8}$$

which can be written in the form

$$z = Sz_0 + LNz(\cdot) \quad , \quad z(0) = z_0 \tag{3.2.9}$$

where $L$ and $S$ are given by (2.2.32) and (2.2.33) respectively and $z = (z_0, z(\cdot))$, the pair initial state-trajectory, lies in a space $X$ of the type $M(0,T;Z)$ (e.g. $X = M^p(0,T;Z)$, $p = 0$ or $1 \le p < \infty$).

The output equation is (3.2.4), namely

$$y_e = Cz(\cdot) \quad . \tag{3.2.10}$$

Similarly to linear systems we define $z = (z_0, z(\cdot)) \in X$ to be an estimated state if $z$ satisfies (3.2.9) and the output equation (3.2.10). For a given $\varepsilon \ge 0$ we say that $z = (z_0, z(\cdot)) \in X$ is an $\varepsilon$-estimated state if it satisfies (3.2.9) and

$$Cz(\cdot) \in \overline{B_\varepsilon(y_e)} \quad \text{in } Y \ .$$

Again, here we allow a possible error $\varepsilon$ in the measurements. Clearly if we set $\varepsilon = 0$ then

$z$ is an $\varepsilon$-estimated state $\Rightarrow$ $z$ is an estimated state.

## 3.3 - STATE AND PARAMETER ESTIMATION.

In this section we consider the joint problem of state estimation and parameter identification for a system

$$\dot{z}(t) = \mathcal{A}z(t) + A_1\alpha + N(z(t),\alpha) \ , \quad z(0) = z_0 \in Z$$
$$y_e(t) = Cz(t) \ . \tag{3.3.1}$$

where the operators $\mathcal{A}$ and $C$ are defined as in section 3.1, $A_1 : Z_1 \to Z$, the nonlinearity $N \in \mathcal{O}p(\mathfrak{F}(D',Z) \times Z_1, \mathfrak{F}(D',Z))$, $D' = [0,T]$ and $Z_1$ is a finite dimensional Banach space of parameters $\alpha$ with $\dim(Z_1) = p$ (e.g., $Z_1 = \mathbb{R}^p$). System (3.3.1) may be derived from the linearisation of a system described by

$$\dot{z}(t) = f(z,u,\alpha,t) \ , \quad z(0) = z_0$$
$$y_e(t) = h(z,u,\alpha,t) \ .$$

The joint problem of state and parameter estimation is to construct the state $z(t)$, $t \in [0,T]$ and identify the parameters $\alpha \in Z_1$ for system (3.3.1) when the observation $y_e \in \mathcal{Y}$ is given. Observe that if we take system (3.3.1) without the nonlinearity $N$ we have the system

$$\dot{z}(t) = \mathcal{A}\, z(t) + A_1\alpha \ , \qquad z(0) = z_0 \in Z$$
$$y_e(t) = Cz(t) \tag{3.3.2}$$

which has mild solution

$$z(t) = S(t)z_0 + \int_0^t S(\tau)A_1\alpha \, d\tau \ , \quad z(0) = z_0 \in Z$$
$$y_e(t) = Cz(t) \ . \qquad (3.3.3)$$

Since the parameter $\alpha \in Z_1$ is assumed to be constant, the following equation can be added to (3.3.1)

$$\dot{\alpha} = 0$$

and hence (3.3.1) becomes

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} A & A_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z(t) \\ \alpha \end{bmatrix} + \begin{bmatrix} N(z(t),\alpha) \\ 0 \end{bmatrix} , \begin{bmatrix} z \\ \alpha \end{bmatrix} (0) = \begin{bmatrix} z_0 \\ \alpha \end{bmatrix} \qquad (3.3.4)$$
$$y_e(t) = [C \quad 0] \begin{bmatrix} z(t) \\ \alpha \end{bmatrix}$$

which we call the overall system for (3.3.1). If we denote

$$\bar{z}(t) = \begin{bmatrix} z(t) \\ \alpha \end{bmatrix} \quad \bar{z}_0 = \begin{bmatrix} z_0 \\ \alpha \end{bmatrix} \ , \ \bar{Z} = Z \times Z_1 \ ,$$
$$\bar{A} = \begin{bmatrix} A & A_1 \\ 0 & 0 \end{bmatrix} \ , \quad \bar{N}(z(\cdot),\alpha) = \begin{bmatrix} N(z(\cdot),\alpha) \\ 0 \end{bmatrix}$$

and

$$\bar{C} = [C \quad 0]$$

then the overall system (3.3.4) becomes

$$\dot{\bar{z}}(t) = \bar{A}\,\bar{z}(t) + \bar{N}\,\bar{z}(t)\ , \qquad \bar{z}(0) = \bar{z}_0 \in \bar{Z}$$
$$y_e(t) = \bar{C}\,\bar{z}(t) \tag{3.3.5}$$

and $\bar{A}$ generates the strongly continuous semigroup $\bar{S}(\cdot)$ given by

$$\bar{S}(t)\bar{z}_0 = \begin{bmatrix} S(t) & L_1(t) \\ 0 & I_1 \end{bmatrix} \begin{bmatrix} z_0 \\ \alpha \end{bmatrix}$$

where $I_1$ is the identity on $Z_1$ and $L_1(t) : Z_1 \to Z$

$$L_1(t)\alpha = \int_0^t S(\tau)A_1\alpha\, d\tau \quad .$$

So the joint problem of state and parameter estimation of (3.3.1) is transformed in state estimation of (3.3.5) and the techniques developed in chapter 7 for state estimation will then have an immediate extension for problems of state and parameter estimation of systems of this type.

In the sequel we present some results obtained for system (3.3.1) which will have some application in section 7.2.

### 3.3.1 - Overall observability.

Observe that the linear part of (3.3.5) has the mild solution

$$\bar{z}(t) = \bar{S}(t)\bar{z}_0 \quad , \quad \bar{z}(0) = \bar{z}_0 \in \bar{Z}$$
$$y_e(t) = \bar{C}\,\bar{z}(t) \quad . \tag{3.3.6}$$

Basically equations (3.3.3) and (3.3.6) are the same, howevever when we refer to the overall system (3.3.6) we mean a system with state space $\overline{Z} = Z \times Z_1$ whereas we regard (3.3.3) as a system with state space $Z$ and a perturbation.

We say that (3.3.3) is initially (continuously initially) overall observable if system (3.3.6) is initially (continuously initially) observable.

Clearly if we have initial overall observability of (3.3.3) then $\overline{CS}(\cdot):\overline{Z} \to \mathcal{Y}$ is a one-to-one mapping and this clearly implies that $CS(\cdot):Z \to \mathcal{Y}$ is also a one-to-one mapping and therefore we have initial observability of (3.3.3). It is also immediate to verify that continuous initial overall observability of (3.3.3) implies continuous initial observability of (3.3.3).

Suppose now that $A_1$ is not injective, i.e., $N(A_1) \neq \{0\}$. Thus we can find $\alpha_0 \in Z_1$, $\alpha_0 \neq 0$ such that $A_1\alpha_0 = 0$ and for $\overline{z}_0 = (0,\alpha_0) \neq 0$ we have

$$\overline{CS}(\cdot)\overline{z}_0 = CL(\cdot)A_1\alpha_0 = 0$$

which implies that $N(\overline{CS}(\cdot)) \neq \{0\}$, that is, the overall system cannot be either initially nor continuously initially observable.

Summarizing these two necessary conditions for overall observability we have

3.3.2 - Lemma.

If system (3.3.3) is initially (continuously initially) overall observable, then

i) system (3.3.3) is initially (continuously initially) observable;

ii) $A_1$ is a one-to-one mapping. (3.3.7)

□

Actually if (3.3.7) does not hold we have more parameters than really needed in the model since we can redefine $\alpha$ in the new space of parameters $\hat{Z}_1 = Z_1/N(A_1)$ of smaller dimension than $Z_1$. If $[\alpha] \in \hat{Z}_1$ is the equivalent class of $\alpha + \tilde{\alpha}$ for which $A_1(\tilde{\alpha}) = 0$ and $\hat{A}_1 : \hat{Z}_1 \to Z$ is defined by

$$\hat{A}_1[\alpha] = A_1\alpha$$

then $\hat{A}_1$ is a one-to-one mapping. We can easily give a norm to $\hat{Z}_1$ such that it will also be a finite dimensional Banach space (e.g., take $\hat{\alpha} = P(\alpha)$ to be the representative of the class $[\alpha] \in \hat{Z}_1$ where $P$ is any linear projection satisfying $N(P) = N(A_1)$. Hence

$$\|[\alpha]\|_{\hat{Z}_1} = \|\hat{\alpha}\|_{Z_1}$$

gives the wanted structure for $(\hat{Z}_1, \|\cdot\|_{\hat{Z}_1}))$.

In the sequel we shall assume that $A_1$ satisfies (3.3.7), i.e.,

$$N(A_1) = \{0\} \quad .$$

Consider now the following theorem which establishes another necessary condition for overall observability

3.3.3 - Theorem.

Let $\Theta \in \mathcal{LO}p(\overline{Z}, Z\times Y)$ be given by

$$\Theta = \begin{bmatrix} A & A_1 \\ C & 0 \end{bmatrix} \quad . \tag{3.3.8}$$

If system (3.3.3) is initially overall observable then

$$\mathcal{N}(\Theta) = \{0\} \ . \tag{3.3.9}$$

Proof.

Assume that (3.3.3) is initially overall observable but (3.3.9) does not hold, that is $\mathcal{N}(\Theta) \neq \{0\}$. Hence, there is $(z_0,\alpha) \in \mathcal{D}(A) \times Z_1$, $(z_0,\alpha) \neq 0$, such that

$$Az_0 + A_1\alpha = 0 \tag{3.3.10}$$

$$Cz_0 \qquad = 0 \ . \tag{3.3.11}$$

Acting $C\mathcal{S}(\cdot)$ on both sides of (3.3.10) and integrating on $[0,t]$, $t \geq 0$, we get

$$\int_0^t C\mathcal{S}(\tau)z_0 d\tau \ - Cz_0 + \int_0^t C\mathcal{S}(\tau)A_1\alpha \ d\tau = 0, \ \forall \ t \geq 0 \quad .$$

Thus, since $z_0 \in \mathcal{D}(A)$ we can write

$$CS(t)z_0 - Cz_0 + \int_0^t CS(\tau)A_1\alpha \, d\tau = 0 \quad \forall\, t \geq 0 \tag{3.3.12}$$

and hence, substituting (3.3.11) in (3.3.12) we obtain

$$\overline{CS}(t)\ (z_0,\alpha) = 0 \qquad \forall\, t \geq 0$$

and therefore, since $\mathcal{D}(A)$ is dense in $Z$ ,

$$N(\overline{CS}(\cdot)) \neq 0$$

which is a contradiction to the assumption of overall observability. So (3.3.9) must hold.

Q.E.D.

We shall now consider systems with finite dimensional state space $Z$ . Suppose that $\dim(Z) = n$ (i.e. $n$ states), $\dim(Z_1) = p$ (i.e., $p$ parameters) and $\dim\,(Y) = q$ (i.e., $q$ outputs). We can assume, without loss of generality, that

$$Z = \mathbb{R}^n\ , \quad Z_1 = \mathbb{R}^p \quad \text{and} \quad Y = \mathbb{R}^q\ .$$

Also suppose that $\overline{A}$, $A$, $A_1$ and $C$ are respectively the $(n+p \times n+p)$, $(n \times n)$, $(p \times n)$ and $(n \times q)$ matrices representation of these operators.

Now the concepts of initial (overall) observability and continuous initial (overall) observability coincide we shall call them both (overall) observability.

Observe that $\Theta$ defined in (3.3.8) is now a $(n+q \times n+p)$ matrix. Consider the following theorem

3.3.4 - Theorem.

System (3.3.3) is overall observable if and only if

(a) system (3.3.3) is observable;

(b) rank $(\Theta) = n + p$

where $\Theta$ is given by (3.3.8).

Proof.

Necessity: Assume (3.3.3) is overall observable then, by lemma 3.3.2 we have that (a) holds and by theorem 3.3.3 we have that (b) holds.

Sufficiency: Assume that (a) and (b) hold but (3.3.3) is not overall observable. Thus,

$$y_e(t) = CS(t)z_0 + \int_0^t CS(s)A_1\alpha \, ds = 0 , \quad \forall \, t \geq 0 \tag{3.3.13}$$

for some $(z_0,\alpha) \neq (0,0)$, $(z_0,\alpha) \in Z \times Z_1$. Setting $t = 0$ we have

$$Cz_0 = 0 \tag{3.3.14}$$

and from (3.3.13) we can also write

$$\frac{dy_e(t)}{dt} = CS(t)Az_0 + CS(t)A_1\alpha = 0 \quad , \quad \forall\, t \geq 0$$

and hence,

$$CS(\cdot)Az_0 + CS(\cdot)A_1\alpha = 0 \; . \tag{3.3.15}$$

By (a) we have that $CS(\cdot)$ has left inverse and therefore (3.3.15) becomes

$$Az_0 + A_1\alpha = 0 \; . \tag{3.3.16}$$

If we set $\bar{z} = (z_0,\alpha)$ then (3.3.14) and (3.3.16) imply that

$$\Theta\bar{z} = 0 \quad \text{for some} \quad \bar{z} \neq 0$$

which is a contradiction to assumption (b). So, when (a) and (b) hold system (3.3.3) must be overall observable. This concludes the proof.

Q.E.D.

3.3.5 - <u>Corollary</u>.

A necessary condition for overall observability is

$$q \geq p$$

(i.e., the number of outputs is not less than the number of parameters).

Proof.

Immediate from theorem 3.3.4.

□

A standard result in finite dimensional linear system theory is that system (3.3.6) is observable if and only if

$$\operatorname{rank}(\mathcal{O}) = n + p \tag{3.3.17}$$

where $\mathcal{O}$ is the observability matrix for system (3.3.6) given by

$$\mathcal{O} = [\bar{C}' \quad \bar{A}'\bar{C}' \quad \dots \quad \bar{A}'^{(n+p-1)}\bar{C}'] \;. \tag{3.3.18}$$

Clearly (3.3.17) gives necessary and sufficient conditions for overall observability of (3.3.3) which does not require to check observability of (3.3.3). We shall see in theorem 3.3.7 that condition (3.3.17) has a simpler form when $\mathcal{O}$ is given by (3.3.18). First we see the following result:

3.3.6 - Lemma.

Let $\Delta^A(\lambda)$ be the characteristic polynomial of $A$, then

$$\bar{A}\,\Delta^A(\bar{A}) = 0 \;.$$

Proof.

Let $\gamma_1, \gamma_2, \dots, \gamma_n$ be the coefficients of the characteristic polynomial of $A$ , i.e.,

$$\Delta^A(\lambda) = \det(\lambda I_n - A) \equiv \lambda^n + \gamma_1\lambda^{n-1} + \ldots + \gamma_{n-1}\lambda + \gamma_n \qquad (3.3.19)$$

where $I_n$ is the identity on $\mathbb{R}^n$ . Setting $\gamma_0 = 1$ , (3.3.19) becomes,

$$\Delta^A(\lambda) = \sum_{i=0}^{n} \gamma_i \lambda^{n-i}$$

and hence

$$\Delta^A(\bar{A}) = \sum_{i=0}^{n} \gamma_i \bar{A}^{n-i} \quad .$$

Now, since $\bar{A}^k = \begin{bmatrix} A^k & A^{k-1}A_1 \\ 0 & 0 \end{bmatrix}$ for all $k \geq 1$ and

$\bar{A}^0 = I_{n+p} = \begin{bmatrix} I_n & 0 \\ 0 & I_p \end{bmatrix}$ we obtain

$$\Delta^A(\bar{A}) = \begin{bmatrix} \sum_{i=0}^{n} \gamma_i A^{n-i} & (\sum_{i=0}^{n-1} \gamma_i A^{n-i-1} A_1) \\ 0 & \gamma_n I_p \end{bmatrix}$$

$$= \begin{bmatrix} \Delta^A(A) & (\sum_{i=0}^{n-1} \gamma_i A^{n-i-1}) A_1 \\ 0 & \gamma_n I_p \end{bmatrix} \quad .$$

Therefore, computing $\bar{A}\Delta^A(\bar{A})$ we get

$$\bar{A}\Delta^A(\bar{A}) = \begin{bmatrix} \Delta^A(A)A & \Delta^A(A)A_1 \\ 0 & 0 \end{bmatrix}$$

and applying Cayley-Hamilton theorem (i.e., $\Delta^A(A) = 0$) , we complete the proof.

Q.E.D.

3.3.7 - Theorem.

System (3.3.3) is overall observable if and only if

$$\text{rank } (\mathcal{O}) = n + p$$

where $\mathcal{O}$ is the matrix formed by the first $(n+1)q$ columns of the matrix $\mathcal{O}$ in (3.3.18), that is

$$\mathcal{O} = \begin{bmatrix} C' & A'C' & A'^2C' & \ldots A'^nC' \\ 0 & A_1'C' & A_1'A'C' & \ldots A_1'A'^{n-1}C' \end{bmatrix} .$$

Proof.

Using lemma 3.3.4 we can write

$$\bar{A}(\bar{A}^n + \gamma_1\bar{A}^{n-1} + \ldots + \gamma_{n-1}\,\bar{A} + \gamma_n I_{n+p}) = 0$$

where $\gamma_i$ , $1 \le i \le n$ are the coefficients of $\Delta^A(\lambda)$ , the characteristic polynomial of $A$ . Thus,

$$\bar{A}^{n+1} + \gamma_1\bar{A}^n + \ldots + \gamma_{n-1}\bar{A}^2 + \gamma_n\bar{A} = 0$$

which means that $\bar{A}^{n+1}$ , and so $\bar{A}^{n+k}$ for all $k \geq 1$ , can be expressed as a linear combination of $\bar{A}^n$ , $\bar{A}^{n-1}, \ldots, \bar{A}$ . Therefore the columns of the matrix

$$[\bar{A}'^{\,n+1}\bar{C}' \quad \bar{A}'^{\,n+2}\bar{C}' \quad \ldots \quad \bar{A}'^{\,n+p-1}\bar{C}'] \tag{3.3.20}$$

can be written as a linear combination of the columns of $\bar{\mathcal{O}}$ . But matrix (3.3.20) are exactly the last $q(p-1)$ columns of $\mathcal{O}$ defined in (3.2.18) and therefore condition (3.2.17) (i.e., rank $\mathcal{O} = n+p$) can be relaxed to become rank $\bar{\mathcal{O}} = n + p$ and we end the proof.

Q.E.D.

3.3.8 - Ending remark.

Observe that

$$(\text{columns of } \mathcal{O}) - (\text{columns of } \bar{\mathcal{O}}) = (p-1)q \; .$$

In the case $p = 1$ (only one parameter), $\bar{\mathcal{O}}$ coincides with $\mathcal{O}$ . Otherwise $\bar{\mathcal{O}}$ is a smaller matrix.

□

# CHAPTER 4.

## PROJECTIONS.

In this chapter we present some classical results about projections as well as bring some new concepts related with projections which will have applications in later chapters. Projection here has the broad definition of an idempotent operator and in our treatment we usually associated a projection $P$ with its range $S = \mathcal{R}(P)$ . Linear projections and orthogonal projections are particular cases treated in sections 4.2 and 4.3 respectively.

Among the new concepts we have: "semi-characteristic" and "characteristic functional" for a set $S$ , $\phi_S$ (introduced in example 4.1.8), "active mappings" $Q$ under a projection $P$ (introduced in section 4.5) and "uniform projections" (introduced in section 4.7).

In section 4.4 we show that an orthogonal projection $P$ maps open sets of $X$ into open sets of $\mathcal{R}(P)$ as well as closed and bounded sets into closed and bounded sets. Unfortunately the author could not find reference for these two simple statements and the full proof is carried out here. These results will be extended to linear operators with closed range in chapter 5, a result which will be used in applications to control theory in chapter 6.

## 4.1 - PROJECTIONS.

In the present treatment we usually associated each projection $P$ to a set $S = \mathcal{R}(P)$ .

### 4.1.1 - Definition of Projection.

A mapping $P: X \to X$ is said to be a projection onto $S = \mathcal{R}(P)$ if $P^2 = P$ (i.e., $P$ is idempotent).

We shall always assume that $S$ is non-empty. Clearly any non-empty set $S \subset X$ has a projection $P$ (e.g., $P$ defined by $Px = x$ if $x \in S$ and $Px = x_0$ for some $x_0 \in S$ if $x \notin S$) , though $P$ may be noncontinuous. Elementary examples of continuous projections $P$ onto $S$ are: $P = I$ (identity) for $S = X$ and $Px = x_0$ for $S = \{x_0\}$ .

We shall call $P$ a "linear projection" if $P$ is a projection and a linear operator in $X$ (see sections 4.2 and 4.3). In the particular case $P$ is a closed linear operator densely defined in $X$ (see §4.2.8) we call "closed projection".

### 4.1.2 - A Property.

Every projection $P$ onto $S \subset X$ , independent of being linear or even continuous, satisfies

$$S = \mathcal{N}(I - P) = \{x \in X : Px = x\} .$$

□

### 4.1.3 - Theorem.

If $P: X \to X$ is a continuous projection (not necessarily linear) onto $S \subset X$ , then $S$ is closed.

Proof.

Follows immediately from property 4.1.2 since $P$ continuous implies $(I-P)$ continuous.

□

This result is also valid in a more general context of topological vector spaces (see [37]).

The converse of this theorem is obviously not true, however if $S$ is compact we have the following result:

4.1.4 - Theorem.

If $P$ is a projection onto a compact set $S \subset X$ , then $P$ is a compact operator. If in addition $P$ is continuous, then $P$ is completely continuous.

Proof.

Follows immediately from definitions of projection, compact operators and completely continuous operators (see §2.2.1).

4.1.5 - Examples of projections.

i) This is a classical example of a continuous projection $P:X \to X$ into $S = \overline{B_a(0)}$ = the closed ball of radius $a$ centred at the origin:

$$Px = \begin{cases} x & \text{if } \|x\| \le a \\ \frac{ax}{\|x\|} & \text{if } \|x\| > a \quad . \end{cases} \tag{4.1.1}$$

Observe that if $x \notin S \Rightarrow Px \in \partial S = \{x \in X : \|x\| = a\}$ . $P$ in (4.1.1) is called a-radial retraction mapping [26].

ii) If $S'$ is a closed, convex subset of $X$ such that $0 \in \text{Int}(S')$ then a generalization of the a-radial retraction is given by the continuous mapping $P':X \to X$

$$P'x = \gamma_S(x)\cdot x \tag{4.1.2}$$

where $\gamma_S: X \to \mathbb{R}$ is given by

$$\gamma_S(x) = \begin{cases} 1 & \text{if } x \in S \\ \sup\{k>0 : kx \in S\} & \text{if } x \notin S \; . \end{cases}$$

This is also a classical example. Observe that, similarly to (i), if $x \notin S \Rightarrow P'x \in \partial S$ . □

4.1.6 - <u>Projections onto a closed and convex set of a uniformly convex Banach space</u>.

If $X$ is a uniformly convex Banach space and $S \subset X$ is a closed and convex set then a well defined projection onto $S$ is given by $P:X \to X$

$$Px = \text{the closest element to } x \text{ in } S \tag{4.1.3}$$

that is, $Px \in S$ and

$$\|Px-x\| < \|x-x'\| \quad \forall\, x' \in S\,,\quad x' \neq Px\,.$$

This follows since any closed and convex set in $X$ has a unique element of minimal norm [26], a result which has been widely used in optimization, specially in cases where $X$ is a Hilbert space [4, 23].

Note that $Px \in \partial S$ if $x \notin S$.

4.1.7 - Theorem.

$P$ given by (4.1.3) is continuous.

Proof.

See §1.3 of [26].

□

4.1.8 - More examples of projections.

i) Let $S$ be a closed and convex set with $0 \in \mathrm{Int}(S)$ and $P'$ be the projection onto $S$ given by either (4.1.2) or (4.1.3). Consider $P:X \to X$ given by

$$Px = \frac{P'x}{\phi_S(x)} \tag{4.1.4}$$

where $\phi_S(x) = 1 + \|x - P'x\|$. Clearly $P$ is also a continuous projection onto $S$. Observe that now $Px \in \mathrm{Int}(S)$ for any $x \notin S$. One can easily see that $P$ in (4.1.4) has the property: the further $x$ is from $S$, the closer $Px$ is to the origin.

ii) We can give a more general form for $P$ in (4.1.4). Suppose $\phi_S : X \to \mathbb{R}$ is a continuous functional with the properties

$$\phi_S(x) \geq 1 \tag{4.1.5}$$

and

$$\phi_S(x) = 1 \quad \text{if} \quad x \in S . \tag{4.1.6}$$

We shall call $\phi_S$ a "semi-characteristic functional" for $S$.

Let $P' : X \to X$ be any continuous projection onto $S \subset X$, then $P : X \to X$ given by either

$$Px = P'\left(\frac{x}{\phi_S(x)}\right) \qquad \text{or} \qquad Px = P'(\phi_S(x)x) \tag{4.1.7}$$

are two new continuous projections onto $S$. If $S$ is convex and $0 \in S$ then, if $x \in S$, $kx \in S$ for all $0 \leq k \leq 1$ and we have that

$$Px = \frac{P'x}{\phi_S(x)} \; , \quad Px = \frac{1}{\phi_S(x)} . P'\left(\frac{x}{\phi_S(x)}\right) \; , \; Px = \frac{P(\phi_S(x)x)}{\phi_S(x)} \; , \text{ etc.} \tag{4.1.8}$$

are also continuous projections onto $S$. Moreover, if $S$ is a subspace of $X$ then

$$Px = \phi_S(x)P'x \; , \; Px = \phi_S(x)P'(\phi_S(x)x), \; Px = \phi_S(x)P'\left(\frac{x}{\phi_S(x)}\right), \text{ etc.} \tag{4.1.9}$$

are also continuous projections onto $S$ . Further projections could still be found by using more than one semi-characteristic functional $\phi_S$ .

We call $\phi_S$ a "characteristic functional" for $S$ if (4.1.5) holds and

$$\phi_S(x) = 1 \iff x \in S \ . \tag{4.1.10}$$

Some examples of characteristic functionals are given by

$$\phi_S(x) = m - (m-1)\ \gamma_S(x) \tag{4.1.11}$$

for any $m > 1$ and $\gamma_S$ given by (4.1.2), and also by

$$\phi_S(x) = \frac{m\|P'x-x\| + 1}{\|P'x-x\| + 1} \tag{4.1.12}$$

for any $m > 1$ and any continuous projection $P'$ onto $S$ .

Both $\phi_S$ in (4.1.11) and (4.1.12) satisfy

$$1 \le \phi_S(x) < m \ .$$

This example shows the variety of projections onto $S$ we can generate once we have one projection $P'$ onto $S$ . We shall see more applications of characteristic functionals in §4.2.7 and in section 4.7.

Note that if $\phi_S'$ and $\phi_S''$ are two semi-characteristic functionals for $S$ then so is $\phi_S$ given by either

$$\phi_S(x) = \phi_S'(x)\ \phi_S''(x)$$

or

$$\phi_S(x) = \tfrac{1}{2}[\phi_S'(x) + \phi_S''(x)] \quad .$$

Also, $\phi_S$ will be a characteristic functional for $S$ if either $\phi_S'$ or $\phi_S''$ (or both) is a characteristic functional for $S$ .

iii) Suppose that $S$ is such that

$$S \subset X_S \subseteq X$$

for some closed subspace $X_S$ of $X$ . This is the case if, for example, $S \subset R(T)$ for some linear operator $T:U \to X$ which has closed range. If $P':X_S \to X_S$ and $\overline{P}:X \to X$ are continuous projections onto $S$ and $X_S$ respectively, then $P:X \to X$ defined by

$$P = P'\overline{P} \tag{4.1.13}$$

is clearly a continuous projection onto $S$ .

An application of $P$ in (4.1.13) is in cases where

$$S = T(U_{ad})$$

for some closed, convex and bounded subset $U_{ad} \subseteq U$ . We shall see in chapter 5 that if $U$ is a Hilbert space, since $U_{ad}$ is closed and bounded then $S$ is closed (and bounded too). But $\mathcal{U}_{ad}$ is also convex so $S$ is closed and convex and therefore we could set $P':X_S \to X_S$, $X_S = R(T)$, the projection given by (4.1.3).

If in addition $\mathcal{U}_{ad}$ is symmetric (or any set which contains the origin of $\mathcal{U}$ in its interior), then $S$ as a subset of $(X_S, \|\cdot\|_X)$ will contain the origin in its interior for some subspace $X_S \subseteq R(T)$ and $P':X_S \to X_S$ could also be the projection onto $S$ given by (4.1.2).

iv) If $P_1$ and $P_2$ are two continuous projections onto $S$ then it is easy to verify that

$$P = \tfrac{1}{2}(P_1 + P_2) \tag{4.1.14}$$

is also a continuous projection onto $S$. Similarly, if $P_1,\ldots,P_n$ are $n$ continuous projections onto $S$, then

$$P = \frac{1}{n}\sum_{i=1}^{n} P_i \tag{4.1.15}$$

is a continuous projection onto $S$ as well.

□

4.1.9 - Projection onto translated sets.

If $P:X \to X$ is a continuous projection onto $S$ and $S'$ is the set

$$S' = \overline{x} + S \tag{4.1.16}$$

for a fixed $\overline{x} \in X$, then $P':X \to X$ given by

$$P'x = \bar{x} + P(x - \bar{x}) \tag{4.1.17}$$

is a continuous projection onto $S'$ .

One application of $P'$ in (4.1.17) is in cases where $0 \notin S'$ (or $0 \notin \text{Int}(S')$) . Then, for some $\bar{\bar{x}} \in S'$ (respectively $\bar{\bar{x}} \in \text{Int}(S')$) , we define

$$S = -\bar{\bar{x}} + S' \quad .$$

Clearly, $0 \in S$ (respectively $0 \in \text{Int}(S)$) and for any projection $P$ onto $S$ , $P'$ given by (4.1.17) with $\bar{x} = -\bar{\bar{x}}$ is a projection onto $S'$ .

If $P_\varepsilon : X \to X$ is any continuous projection onto the closed ball $\overline{B_\varepsilon(0)}$ (e.g. the $\varepsilon$-radial retraction given by (4.1.1)) then, $P_d : X \to X$

$$P_d x = x_d + P_\varepsilon(x - x_d) \tag{4.1.18}$$

is a continuous projection onto the closed ball $\overline{B_\varepsilon(x_d)}$ .

4.1.10 - Projections defined in $D \subseteq X$ .

In definition 4.1.1 a projection $P$ is a mapping from a vector space $X$ into itself. One could also consider projections $P$ defined in $D \subseteq X$ .

A mapping $P : D \to D$ , $D \subseteq X$ is said to be a projection onto $S = R(P)$ if $P^2 = P$ .

Note that if $P : X \to X$ is a projection onto $S$ then $P/D$ is not necessarily a projection. However, if $P$ maps $D$ into $D$ then

$P/D:D \to D$ is a projection onto $S \cap D$. Also, if $D \supset S$ then $P/D:D \to D$ maps $D$ into $D$ and is a projection onto $S$.

4.1.11 - Remark.

If $D \cap S$ is nonempty and also $\neq \{x_0\}$, that is, $D \cap S$ has more than one element, then a projection $P$ onto $S$ cannot be a contraction in $D$ since for $x,y \in D \cap S$, with $x \neq y$, one has that $\|Px - Py\| = \|x - y\|$. In particular, if $S$ is a nonempty set with at least two elements, $P$ is not a contraction in $X$.

For the same reason $P$ is never a generalized contraction, nor a nonlinear contraction either.

In some cases $P$ may be a non-expansive mapping and one example of this is the projection $P$ in (4.1.3). Further examples will be provided by orthogonal projections in section 4.3. $P$ can also be a locally almost non-expansive mapping (see definition in §2.5.1).

$\square$

4.1.12 - Example.

Let $P:X \to X$ be continuous projection onto $S \subset X$ and suppose that for some $p \geq 1$ $P$ satisfies

$$\|Px\| \leq p\|x\| \qquad \text{for all} \quad x \in X. \tag{4.1.19}$$

(Obviously if $S$ is nonempty, $p$ could never be $< 1$).

Now, for some $D$ such that $S \subset D \subseteq X$, let $N:D \to X$ be a nonlinear mapping. It is easy to verify that the following relations hold:

i) If $N$ is a $k$-contraction in $D$ and $kp < 1$, then $PN$ is a $kp$-contraction in $D$.

ii) If $N$ is a non-expansive mapping in $D$ and $p = 1$, then $PN$ is non-expansive in $D$.

Similar conditions could be obtained for $PN$ to be a generalized contraction, a nonlinear contraction, etc. □

## 4.2 - LINEAR PROJECTIONS IN NORMED VECTOR SPACES.

Linear projections were defined in §4.1.1 as projections which are linear operators in $X$. This includes the cases where $P$ is continuous (or bounded), completely continuous (or compact) and closed (densely defined) operators in $X$.

### 4.2.1 - Properties of Linear Projections.

If $P:X \to X$ is a linear projection onto $S$, then

i) $S$ is a subspace of $X$;

ii) $(I-P)$ is a linear projection onto $S = \mathcal{R}(I-P)$;

iii) $\mathcal{N}(P) = \mathcal{R}(I-P)$;

iv) $S \cap \mathcal{N}(P) = \{0\}$.

If, in addition, $P$ is continuous, then

v) $S + \mathcal{N}(P) = X$;

vi) $S$ is a closed subspace of $X$.

Proof.

i) Obvious.

ii) Clearly $(I-P)$ is linear and $(I-P)^2 = (I-P)$ .

iii) Immediate by applying property 4.1.2 to the projection $(I-P)$ .

iv) Follows since $x \in S \cap N(P) \Rightarrow x = Px$ and $Px = 0$ .

v) For any $x \in X$ write $x = Px + (I-P)x$ . Clearly $Px \in S$ and $(I-P)x \in R(I-P) = N(P)$ by (iii).

vi) This is a particular case of theorm 4.1.3.

Q.E.D.

4.2.2 - Complemented Subspaces.

A closed subspace $S$ in a Banach space $X$ is said to be complemented if there exists a closed subspace $S^c$ of $X$ such that

$$S \cap S^c = \{0\}$$

and

$$S + S^c = X .$$

In this case $X$ is said to be the direct sum of $S$ and $S^c$ , represented by

$$X = S \oplus S^c$$

and every $x \in X$ can be uniquely expressed by

$$x = x_1 + x_2 \tag{4.2.1}$$

with $x_1 \in S$ and $x_2 \in S^c$ .

Observe that if $X$ is complete and $P:X \to X$ is a continuous linear projection onto a set $S \subset X$ then (by properties 4.2.1(iv) and (v)), $S$ is complemented and

$$X = S \oplus N(P) .$$

Now let $S$ be complemented and define $P:X \to X$

$$Px = x_1 \tag{4.2.2}$$

where $x_1$ is given by (4.2.1), $P$ is a projection onto $S$ .

4.2.3 - Theorem.

If $S$ is a complemented subspace of a Banach space $X$ then there exists a continuous linear projection $P$ onto $S$ and is given by (4.2.2).

Proof.

The existence of $P$ is obvious. To see that $P$ in (4.2.2) is continuous we refer to §5.16 of [37].

□

4.2.4 - Corollary.

A closed subspace $S$ of a Banach space $X$ is complemented if and only if there exists a continuous linear projection $P$ onto $S$ .

Proof.

Follows from theorem 4.2.3 and properties 4.2.1(iv) and (v).

□

4.2.5 - Theorem.

If P is a linear projection onto a closed subspace S of a Banach space $X$ and $N(P)$ is closed, then P is continuous.

Proof.

See §4.8 of [38]. □

4.2.6 - Theorem.

A linear projection $P:X \to X$ onto S is completely continuous if and only if S is a finite dimensional subspace of $X$ .

Proof.

Necessity. If P is completely continuous then S must be of finite dimension otherwise we could find an infinite set $\{e_n\}$ of linearly independent elements of $S = R(P)$ with $\|e_n\| = 1$ and since $Pe_n = e_n$ , P would not be completely continuous.

Sufficiency. If S is a finite dimensional subspace of $X$ , then $P(E)$ is compact for any bounded set $E \subset X$ . This implies that P is compact and therefore completely continuous since P is linear.

Q.E.D.

4.2.7 - Example.

Here we show an example which will be useful in applications of state estimation in chapter 7. Let $X = M^o(0,T;Z) = C(0,T;Z) \times Z$ (see §2.2.6), $S(\cdot)$ and $S$ as in (2.2.33) (see §2.2.7) and define $\overline{P}:X \to X$ by

$$\bar{P}x = (x_0, S(\cdot)x_0) = Sx_0 \qquad \forall\, x = (x_0, x(\cdot)) \in X \,. \tag{4.2.3}$$

Clearly $\bar{P}$ is a continuous linear projection onto S given by

$$S = R(S) = \{(x_0, S(\cdot)x_0) \in X : x_0 \in Z\} \,. \tag{4.2.4}$$

This implies, by theorem 4.1.3, that S is a closed subspace of $X$.

Observe that $\bar{P}$ does not take account of the component $x(\cdot) \in C(0,T;Z)$. Now let $\phi_S : X \to \mathbb{R}$ be the functional defined by

$$\phi_S(x) = \phi_S(x_0, x(\cdot)) = \|x(\cdot) - S(\cdot)x_0\| + 1 \,. \tag{4.2.5}$$

Clearly $\phi_S$ satisfies (4.1.5) and (4.1.10) and hence it is a characteristic functional for S (see example 4.1.8(ii)). Take $\bar{P}' : X \to X$

$$\bar{P}'x = \phi_S(x)\, \bar{P}x \tag{4.2.6}$$

then $\bar{P}'$ is a continuous (nonlinear) projection onto S which takes account of the component $x(\cdot) \in C(0,T;Z)$.

Let now $X = M^P(0,T;Z)$, $1 \le p < \infty$. Again $\bar{P}$ in (4.2.3) and $\bar{P}'$ in (4.2.6) are two continuous projections (in $X$) onto S and by theorem 4.1.3 S is closed in $M^P(0,T;Z)$.

□

4.2.8 - Closed Projections.

Uncomplemented subspaces S do exist in some Banach spaces (which are not Hilbert spaces) and in these cases there will be no continuous linear projection onto $S$. Examples of uncomplemented subspaces of $L^p$ type spaces with $p > 1$, $p \neq 2$ were first given by F.J. Murray [27] in 1937. Later, in his paper [28] of 1945, Murray defined the concept of "quasi-complements". A closed subspace $S^c$ is a quasi-complement of a closed subspace $S$ if $S \cap S^c = \{0\}$ and $S + S^c$ is dense in $X$.

Murray [28] showed that every closed subspace of a separable reflexive Banach space $X$ with $X^*$ separable has a quasi-complement. Also, proper quasi-complements are not unique. That is, if $S^c$ is a quasi-complement which is not a complement then there is a proper subset $S' \subset S^c$ which is also a quasi complement and $S^c$ is itself a proper subset of a quasi-complement $S''$. In other words quasi-complements can be expanded or contracted to other quasi-complements.

Clearly any $x \in S + S^c$ where $S^c$ is a quasi-complement of $S$ has the unique representation (4.2.1). The projection $P: X \to X$ onto $S$ given by (4.2.2) is closed and densely defined in $X$ with $\mathcal{D}(P) = S + S^c$.

G.W. Mackey [24] in 1946 generalized Murray's results to separable normed vector space $X$.

So, complemented subspaces are associated with continuous linear projections (see theorem 4.2.3) whereas uncomplemented subspaces are associated with closed projections.

It can also be shown (see §4.8 of [38]) that if $P$ is a closed projection onto $S$ then $S$ is closed.

4.2.9 - Remark.

The following results from the above discussion: If $S \subset X_S$, $X_S$ is a subspace of $X$ and $X_S$ is not dense in $X$ then there exists $X_c \subset X$, $X_c = N(P)$ for some linear projection P onto $X_S$, such that $X_S \cap X_c = \{0\}$ and

$$X_S + X_c \text{ is dense in } X \text{ if } P \text{ is closed} \tag{4.2.7}$$

or

$$X_S + X_c = X \text{ if } P \text{ is continuous.} \tag{4.2.8}$$

Similarly, if P is any closed (respectively continuous) projection onto $X_S$ then $X_c = N(P)$ satisfies $X_S \cap X_c = \{0\}$ and (4.2.7) (respectively (4.2.8)).

□

4.3 - LINEAR PROJECTIONS IN INNER PRODUCT SPACES.

In this section $X$ denotes the inner product space $(X,<,>)$ and $\|\cdot\|$ is the norm generated by $<,>$ .

4.3.1 - Orthogonal Projection.

Let S be a closed subspace of $X$ and $S^\perp$ the orthogonal complement of S (see §2.4.1). A linear projection P onto S is called orthogonal projection onto S if

$$N(P) = S^\perp . \tag{4.3.1}$$

If (4.3.1) does not hold $P$ is called an oblique (linear) projection. It can be shown that if $P$ is an orthogonal projection onto $S$ then $P = P^*$, i.e., $P$ is self-adjoint (see [36, 38, 39]).

Consider the following example: $S = \overline{\mathrm{Span}\{\phi_k\}}$ where $\{\phi_k\}$ is an (infinite or finite) orthonormal set in $X$ (see §2.4.3). $S$ is a closed subspace of $X$. Take the projection $P:X \to X$ onto $S$

$$Px = \sum_k \langle x,\phi_k\rangle \phi_k . \tag{4.3.2}$$

Observe that $x \in N(P) \iff Px = 0 \iff \langle x,\phi_k\rangle = 0$ for all $k \iff x \in S^{\perp}$. So (4.3.1) holds and therefore $P$ in (4.3.2) is an orthogonal projection onto $S$. By the next theorem if $X$ is a Hilbert space then $P$ is unique.

4.3.2 - <u>The Projection Theorem.</u>

Let $S$ be a closed subspace of a Hilbert space $X$. There is only one orthogonal projection $P$ onto $S$ and it can be expressed by (4.1.3). $S$ and $S^{\perp}$ are both complemented and

$$X = S \oplus S^{\perp} .$$

<u>Proof.</u>

For a classical exposition of these results see any of the following references [2, 3, 4, 16, 23, 32, 36].

□

4.3.3 - Corollary.

Let $P$ be the orthogonal projection onto a closed subspace $S \subset X$. Then:

i) $(I-P)$ is the orthogonal projection onto $S^{\perp}$ ;

ii) Both $P$ and $(I-P) \in \mathcal{L}(X)$ ;

iii) $\|P\|_{\mathcal{L}(X)} = \|(I-P)\|_{\mathcal{L}(X)} = 1$ .

Proof.

i) Immediate by property (4.1.2) and definition of orthogonal projection.

ii) Follows from the projection theorem and theorem 4.1.7.

iii) See [32].

□

4.3.4 - Theorem.

Let $\mathcal{U}$ and $X$ be two Hilbert spaces. If $T:\mathcal{U} \to X$ is a linear transformation and $P:\mathcal{U} \to \mathcal{U}$ is the orthogonal projection onto $S = N(T)^{\perp} \subset \mathcal{U}$ , then:

i) $$T = TP \tag{4.3.3}$$

ii) $\tilde{T}:S \to R(T)$ defined by $\tilde{T}u = Tu$ , $u \in S$ is a bijection and therefore the inverse

$$\tilde{T}^{-1} : R(T) \to S \tag{4.3.4}$$

exists.

iii) $S = N(T)^{\perp} = \overline{R(T^*)}$ .

iv) $\overline{T(S)} = \overline{R(T)} = N(T^*)^{\perp}$ .

<u>Proof.</u>

Clearly

$$u = Pu + (I-P)u \qquad \forall\, u \in U \ .$$

Since $(I-P)\,u \in N(T)$ , (4.3.3) follows immediately and hence (i) is proved. This also implies that $\tilde{T}$ is a surjection since for $x \in R(T)$ take $u$ such that $Tu = x \Rightarrow \tilde{T}Pu = TPu = x$ . But $\tilde{T}$ is also an injection since for $u_1, u_2 \in S$ such that $\tilde{T}u_1 = \tilde{T}u_2 = x \Rightarrow (u_1 - u_2) \in N(T) = S^{\perp} \Rightarrow (u_1 - u_2) = 0 \Rightarrow u_1 = u_2$ . So (ii) is proved.

The proof of (iii) and (iv) can be found in any of the following references: [3, 23, 32].

Q.E.D.

4.3.5 - <u>Generalized Inverses.</u>

Let $U$ and $X$ be two Hilbert spaces. If $T: U \to X$ is a linear transformation such that $R(T)$ is closed in $X$ then the generalized inverse of $T$ , denoted by $T^{\dagger}$ , is the linear transformation from $X$ to $U$ uniquely defined by the following three properties: (see [7, 13, 29]):

i) $T^{\dagger}Tu = u$ for all $u \in \overline{R(T^*)} = N(T)^{\perp}$

ii) $T^{\dagger}x = 0$ for all $x \in N(T^*) = R(T)^{\perp}$

iii) If $x_1 \in R(T)$ and $x_2 \in N(T^*) = R(T)^\perp$ then $T^\dagger(x_1+x_2) = T^\dagger x_1 + T^\dagger x_2$ .

The generalized inverse $T^\dagger : X \to U$ of $T$ is an extension of $\tilde{T}^{-1}$ in (4.3.4) to all $X$ . Moreover, $T^\dagger \in \mathcal{L}(X,U)$ (see [13, 23]).

One of the nicest properties of generalized inverse is that if $x \in R(T)$ , then $u = T^\dagger x$ is the element of minimal norm among all $u' \in U$ such that $Tu' = x$ . If however $x \notin R(T)$ then $x' = Tu = TT^\dagger x$ is the closest element to $x$ in $R(T)$ , that is $\|x'-x\| < \|x''-x\|$ , $\forall x'' \in R(T)$ , $x' \neq x''$ . These properties follow from the fact that one can express the orthogonal projection $P:U \to U$ onto $S = N(T)^\perp \subset U$ by

$$P = T^\dagger T ,$$

and the orthogonal projection $\overline{P}:X \to X$ onto $T(S) = R(T) \subset X$ by

$$\overline{P} = TT^\dagger . \tag{4.3.5}$$

4.3.6 - Example.

Here we present an example which will be used in chapter 6. Let $X = M^s(0,T;Z) = H^s(0,T;Z) \times Z$ (see §2.2.6) and $S$ the subspace of $X$ given by

$$S = R(LB) = \{(L(\cdot)Bu, L(T)Bu) \in X : u \in U\} \tag{4.3.6}$$

and suppose $S$ is closed. Then $(LB)^\dagger : X \to U$ is a well defined bounded linear transformation and the orthogonal projection $\overline{P}:X \to X$ onto $S$ is given by

$$\overline{P} = (LB)(LB)^{\dagger} \quad . \tag{4.3.7}$$

Now suppose that $\mathcal{R}(L(\cdot)B)$ is closed in $H^s[0,T;Z]$, then $(L(\cdot)B)^{\dagger}:H^s[0,T;Z] \to \mathcal{U}$ is also well defined and $\overline{P}':X \to X$

$$\overline{P}'(x(\cdot),x_f) = (LB)(L(\cdot)B)^{\dagger}x(\cdot) \tag{4.3.8}$$

is a continuous projection onto $S$.

The projections $\overline{P}$ and $\overline{P}'$ are different, that is, $\overline{P}'$ is oblique. This follows since if $\overline{P}x = LBu$ for some $u \in \mathcal{U}$, then $u$ is such that

$$||LBu - x||_{M^s} = ||L(\cdot)Bu-x(\cdot)||_{H^s} + ||L(T)Bu - x_f||_Z$$

is minimized whereas if $\overline{P}'x = LBu'$ then $u'$ is such that only the first summand $||L(\cdot)Bu' - x(\cdot)||_{H^s}$ is minimized. In fact $\overline{P}'(x(\cdot),x_f) = \overline{P}'(x(\cdot),0)$, $\forall\, x_f \in Z$.

Now suppose $\mathcal{R}(L(T)B)$ is closed in $Z$, so $(L(T)B)^{\dagger}$ is defined and consider the mapping $\Pi_1:X \to X$

$$\Pi_1(x(\cdot),x_f) = (LB)(L(T)B)^{\dagger}x_f \tag{4.3.9}$$

$\Pi_1$ is not a projection onto $S$ since if we take $x = (x(\cdot),x_f) \in S$ with $x(\cdot) \neq 0$ and $x_f = 0$ we have $\Pi_1 x = 0 \neq x$. However, $\Pi_1$ is a continuous linear projection onto the closed subspace $S_1 \subset S$

$$S_1 = \{(L(\cdot)Bu,L(T)Bu) \in X : u \in N(L(T)B)^{\perp}\} \subset S \ . \tag{4.3.10}$$

Now let P be any continuous projection onto S (e.g. $P = \overline{P}$ in (4.3.7) or $P = \overline{P}'$ in (4.3.8). We define $\Pi_2 : X \to X$ by

$$\Pi_2 = (I - \Pi_1)P \quad . \tag{4.3.11}$$

It is easy to check that $(\Pi_2)^2 = \Pi_2$ , i.e., $\Pi_2$ is a continuous projection onto $\mathcal{R}(\Pi_2)$ . Moreover, $\Pi : X \to X$ given by

$$\Pi = \Pi_1 + \Pi_2 \tag{4.3.12}$$

is a continuous projection onto S . If P is linear then $\Pi$ is also linear.

□

4.3.7 - Remark.

By corollary 4.3.3(iii) if P is an orthogonal projection onto a closed subspace S of a Hilbert space then both P and (I-P) satisfy (4.1.19) with $p = 1$ and since orthogonal projections are linear, both P and (I-P) are non-expansive mappings. In fact orthogonal projections are the only non-expansive linear projections. Any oblique (linear) projection P has norm $\|P\| > 1$ and therefore cannot be non-expansive.

It is also clear that orthogonal projections are locally almost non-expansive mappings (see §2.5.1).

□

4.3.8 - Remark.

If $X$ is the Hilbert space $M^2(0,T;Z)$ and $\overline{P} : X \to X$ is given by (4.2.3) then $\overline{P}$ is an oblique projection.

To see this first take

$$x = (x_0, - S(\cdot)x_0) \in X \tag{4.3.13}$$

for some $x_0 \in Z$ , $x_0 \neq 0$ . Thus, for $x = (x_0, x(\cdot))$ as in (4.3.13) one has that the inner-product

$$<x(\cdot), S(\cdot)x_0>_{L^2} = - \|S(\cdot)x_0\|_{L^2} \neq 0 \quad . \tag{4.3.14}$$

Now suppose $\overline{P}$ is an orthogonal projection, thus

$$<(I-\overline{P})x, \overline{P}x>_X = 0 \quad \text{for all} \quad x \in X. \tag{4.3.15}$$

But (4.3.15) is equivalent to

$$<x(\cdot), S(\cdot)x_0>_{L^2} - <S(\cdot)x_0, S(\cdot)x_0>_{L^2} = 0 \quad \text{for all} \quad x \in X ,$$

and hence

$$<x(\cdot), S(\cdot)x_0>_{L^2} = \|S(\cdot)x_0\|_{L^2} \text{ for all } x=(x_0, x(\cdot)) \in X \quad . \tag{4.3.16}$$

Now, taking $x = (x_0, x(\cdot))$ as in (4.3.13), equations (4.3.14) and (4.3.16) show a contradiction. So $\overline{P}$ is not orthogonal.

□

## 4.4 - <u>TWO PROPERTIES OF ORTHOGONAL PROJECTIONS</u>.

First we show that for an orthogonal projection P onto S , if $E \subset X$ is open then P(E) is open in S . Clearly since S is a closed subspace of $X$ , unless $S = X$ , $P(E)$ is never closed in $X$ .

4.4.1 - Theorem.

If $P$ is the orthogonal projection onto a closed subspace $S$ of a Hilbert space $X$ , then $P$ maps open sets of $(X,\|\cdot\|)$ into open sets of $(S,\|\cdot\|)$ .

Proof.

Clearly we only need to show that for any $x_0 \in X$ ,

$$P(B_\varepsilon(x_0)) = \tilde{B}_\varepsilon(Px_0) \tag{4.4.1}$$

where $B_\varepsilon(x_0)$ is the open ball in $X$ of radius $\varepsilon$ and centred at $x_0$ and $\tilde{B}_\varepsilon(Px_0)$ is the open ball in $S$ of radius $\varepsilon$ and centred at $Px_0$ . First set $x_0' = (I-P)x_0$ . Then we can write $x_0$ in the unique decomposition

$$x_0 = Px_0 + x_0'$$

where $Px_0 \in S$ and $x_0' \in S^\perp$ . If we take $x \in \tilde{B}_\varepsilon(Px_0)$ we have that

$$\|x - Px_0\| < \varepsilon \; . \tag{4.4.2}$$

Now set $\overline{x} = x + x_0'$ . Since $x \in S$ and $x_0' \in S^\perp$ ,

$$x = P\overline{x} \tag{4.4.3}$$

and $\|\overline{x} - x_0\| = \|x+x_0' - (Px_0+x_0')\| = \|x-Px_0\|$ . So, by (4.4.2) ,

$$\|\overline{x} - x_0\| < \varepsilon \tag{4.4.4}$$

that is,

$$\overline{x} \in B_\varepsilon(x_0) . \tag{4.4.5}$$

Equations (4.4.3) and (4.4.5) imply that $x \in P(B_\varepsilon(x_0))$ . So,

$$\tilde{B}_\varepsilon(Px_0) \subset P(B_\varepsilon(x_0)) . \tag{4.4.6}$$

Now take $\overline{x} \in B_\varepsilon(x_0) \Rightarrow$ (4.4.4) holds. Set $\tilde{x} = P\overline{x} + x_0'$ . It is easy to check that

$$(\tilde{x} - x_0) = P(\overline{x}-x_0) \in S$$

and

$$[(I-P)\overline{x} - x_0'] = (I-P)(\overline{x} - x_0') \in S^\perp .$$

So $(\tilde{x}-x_0) \perp [(I-P)\overline{x}-x_0']$ and using (2.4.3), Pythagoras theorem, we obtain

$$\|\tilde{x}-x_0\|^2 + \|(I-P)\overline{x}-x_0'\|^2 = \|\tilde{x}-x_0 + \overline{x} - P\overline{x}-x_0'\|^2 = \|\overline{x}_0 - x_0\|^2$$

and therefore, by (4.4.4), we get $\|\tilde{x}-x_0\| < \varepsilon$ which implies that $\tilde{x} \in B_\varepsilon(x_0)$ . Clearly $P\tilde{x} = P\overline{x}$ so ,

$$\|P\overline{x}-Px_0\| = \|P\tilde{x}-Px_0\| = \|\tilde{x}-x_0'-(x_0-x_0')\| = \|\tilde{x}-x_0\| < \varepsilon$$

and hence $P\overline{x} \in \tilde{B}_\varepsilon(Px_0)$ , which implies

$$P(B_\varepsilon(x_0)) \subset \tilde{B}_\varepsilon(Px_0) . \tag{4.4.7}$$

Clearly (4.4.6) and (4.4.7) implies (4.4.1).

Q.E.D.

4.4.2 - Remark.

Theorem 4.4.1 could suggest that $P$ also maps closed sets into closed sets. This is not true and a simple counter-example is given in $X = \mathbb{R}^2$ by

$$E = \{(x, \tan x) : -\frac{\pi}{2} < x < \frac{\pi}{2}\} \subset \mathbb{R}^2 ,$$

and $P$ the orthogonal projection onto $S$ = x-axis = $\{(x,0) : x \in \mathbb{R}\}$ . Clearly $E$ is a closed set, however

$$P(E) = \{(x,0) : -\frac{\pi}{2} < x < \frac{\pi}{2}\} = (-\frac{\pi}{2}, \frac{\pi}{2}) \times \{0\} \subset \mathbb{R}^2$$

is not closed. Nevertheless taking a set $E$ closed and bounded we shall see that $P(E)$ is closed and bounded. The proof is given in section 4.4.4. First we show the following lemma:

□

4.4.3 - Lemma.

If $P$ is the orthogonal projection onto a closed subspace $S$ of a Hilbert space $X$ , then $\{x_n\} \to x$ weakly $\Rightarrow$ $Px_n \to Px$ weakly.

Proof.

Take a sequence $\{x_n\}$ weakly convergent to $x$ , thus

$$\langle x_n, y\rangle \to \langle x, y\rangle \qquad \text{for all} \quad y \in X$$

that is

$$\langle Px_n,y\rangle + \langle (I-P)x_n,y\rangle \to \langle Px,y\rangle + \langle (I-P)x,y\rangle \quad \text{for all } y \in X \tag{4.4.8}$$

and then, since for $y \in S, [(I-P)x_n] \perp y$ and $[(I-P)x] \perp y$, (4.4.8) yields

$$\langle Px_n,y\rangle \to \langle Px,y\rangle \quad \text{for all } y \in S . \tag{4.4.9}$$

Now let $y' \in X$. Thus, $y' = Py' + (I-P)y'$ which implies

$$\langle Px_n,y'\rangle = \langle Px_n,Py'\rangle \tag{4.4.10}$$

$$\langle Px,y'\rangle = \langle Px,Py'\rangle \quad . \tag{4.4.11}$$

Using (4.4.9) we have

$$\langle Px_n,Py'\rangle \to \langle Px,Py'\rangle \quad \text{for all } y \in X . \tag{4.4.12}$$

Substituting (4.4.10) and (4.4.11) in (4.4.12) we obtain

$$\langle Px_n,y'\rangle \to \langle Px,y'\rangle \quad \text{for all } y' \in X$$

which is the same to say that

$$\{Px_n\} \to Px \quad \text{weakly.}$$

Q.E.D.

4.4.4 - Theorem.

If $P$ is the orthogonal projection onto a closed subspace $S$ of a Hilbert space then $P$ maps closed and bounded sets into closed and bounded sets.

Proof.

Take $E$ any closed and bounded set of $X$ . Let $\{z_n\} \in P(E) \subset S$ be a convergent sequence with limit $z$ , that is

$$\{z_n\} \to z \text{ (strongly) and } z_n \in P(E) \subset S \quad \forall n .$$

First we prove that $z \in P(E)$ which implies that $P(E)$ is closed. To see this set $x_n \in E$ with $x_n$ chosen such that

$$Px_n = z_n \in P(E) .$$

Since $E$ is bounded => $\{x_n\}$ is a bounded sequence, by the weak compactness property (see §2.4.10), this implies that there exists a weakly convergent subsequence $\{x_k'\}$ of $\{x_n\}$ . Set $x'$ = the weak-limit of $\{x_k'\}$ . Now, since the weak-closure of a set = closure of the set (see §3.12 of [37]), and since $E$ is closed, one has that $x' \in \overline{E} = E$ and therefore,

$$\{x_k'\} \overset{w}{\to} x' \in E . \tag{4.4.13}$$

Since $\{Px_k'\}$ is a subsequence of $\{z_n\}$ , clearly

$$\{Px_k'\} \overset{w}{\to} z . \tag{4.4.14}$$

Applying lemma 4.4.3 to (4.4.13) and (4.4.14) we obtain,

$$Px' = z$$

and since $x' \in E$ , $z \in P(E)$ . So $P(E)$ is closed.

Clearly $P(E)$ is also bounded since $P \in \mathcal{L}(X)$ and $E$ is bounded.

Q.E.D.

## 4.5 - ACTIVE MAPPING UNDER A PROJECTION.

Here we introduce the concept of "active mapping" under $P$ where $P:X \to X$ is a projection onto a set $S \subset X$ . We shall use these mappings in applications in chapters 6 and 7.

### 4.5.1 - Definition of active mappings.

Let $D \subseteq X$ and $Q:D \to X$ be a mapping. We define $Q$ an active mapping under (the action of) $P$ if

$$Q(x) \neq 0 \Rightarrow Q(x) \notin R(I-P) .$$

If $S$ is a subspace of $X$ and $P$ is linear, then two equivalent definitions could be given by:

$$PQ(x) = 0 \Rightarrow Q(x) = 0$$

or

$$Q(x) \neq 0 \Rightarrow PQ(x) \neq 0 .$$

Also observe that $Q$ is active under $P$ if and only if $R(Q) \cap R(I-P)$ is either $\{0\}$ or $\emptyset$ (the empty set).

### 4.5.2 - Example.

Let $P:X \to X$ be a linear projection onto a closed subspace

$S \subseteq X$ . Then, by property 4.2.1(ii) the range of $(I-P)$ satisfies

$$R(I-P) = N(P)$$

which is a closed subspace of $X$ . Clearly any mapping $Q:D \to X$ such that $Q(x) \notin N(P) \; \forall \, x \in D$ will be an active mapping under $P$ .

□

4.5.3 - Some properties of active mappings.

Let $Q:D \to X$ , $D \subseteq X$

i) $Q$ is always active under $P = I$ (the identity on $X$);

ii) If $Q = 0$ then $Q$ is active under any projection $P$ ;

iii) If $Q \neq 0$ then $Q$ is never active under $P = 0$ ;

iv) If $R(I-P) = X$ then there is no mapping $Q \neq 0$ active under $P$ .

4.5.4 - Remark.

We have seen in example 4.5.2 that it is very easy to find an active mapping $Q$ under $P$ if $P$ is a linear projection. Unfortunately the situation is different when $S$ is a bounded set and $P$ is a continuous projection onto $S$ . In this case, very often we have

$$R(I-P) = X \qquad (4.5.1)$$

and by property 4.5.3 (iv) this implies that we cannot find a mapping $Q$ active under $P$ . (Note that (4.5.1) holds for $P$ in (4.1.1) and, when $S$ is bounded, for $P$ in (4.1.2) and (4.1.3)).

We could get an active mapping $Q$ under $P$ when $S$ is bounded if $P$ is non-continuous (e.g. If $P$ is given by

$$Px = \begin{cases} x & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

then $\mathcal{R}(I-P) = \{0\} \cup X \backslash S$ and any mapping $Q$ such that $Q(x) \in S$ $\forall\, x \in D$ is active under such $P$). However in the applications of chapter 6 and 7 we shall be more interested in continuous projections and therefore we shall use $Q = 0$ in cases where $P$ is not linear.

$\square$

## 4.6 - CHARACTERISTIC AND SEMI-CHARACTERISTIC FUNCTIONALS.

Here we present some examples of characteristic and semi-characteristic functionals for two sets $S$ which will have particular importance in applications in later chapters.

### 4.6.1 - $S = \mathcal{R}(LB)$ .

Consider the set $S$ given by (4.3.6), namely

$$S = \mathcal{R}(LB) = \{(L(\cdot)Bu, L(T)Bu) \in X : u \in \mathcal{U}\} \quad .$$

Let $P:X \to X$ be any projection onto $S$ . (We have seen in example 4.3.7 some examples of projections onto $S$ .) Define $\tau:X \to \mathbb{R}$ by

$$\tau(x) = \|x - Px\|_X \quad .$$

By property 4.1.2 we have

$$\tau(x) = 0 \iff x \in S .$$

Now define $\phi_S : X \to \mathbb{R}$

$$\phi_S(x) = 1 + \tau(x)$$

then $\phi_S$ satisfies both (4.1.5) and (4.1.10) and hence it is a characteristic functional for $S$. Other examples are given by $\phi_S' : X \to \mathbb{R}$

$$\phi_S'(x) = \frac{m\tau(x) + 1}{\tau(x) + 1}$$

for some $m > 1$. In this case $\phi_S'$ satisfies

$$1 \le \phi_S'(x) < m . \tag{4.6.1}$$

Now consider the following notation: $\tilde{x}(\cdot)$ and $\tilde{x}_f$ represent respectively the first and the second components of $Px$, i.e., $Px = (\tilde{x}(\cdot), \tilde{x}_f)$, $\tilde{x}(\cdot) \in H^s(0,T;Z)$ for some $s \in \mathbb{R}$ and $\tilde{x}_f \in Z$ for all $x = (x(\cdot), x_f) \in X = M^s(0,T;Z)$. Define $\theta : X \to \mathbb{R}$ by

$$\theta(x) = \|\tilde{x}(T) - x_f\|_Z \quad \forall\, x = (x(\cdot), x_f) \in X = M^2(0,T;Z) .$$

Then, $\tilde{\phi}_S : X \to \mathbb{R}$ and $\tilde{\phi}_S' : X \to \mathbb{R}$ given by

$$\tilde{\phi}_S(x) = 1 + \theta(x) \tag{4.6.2}$$

$$\tilde{\phi}_S'(x) = \frac{m\theta(x) + 1}{\theta(x) + 1} , \quad m > 1 \tag{4.6.3}$$

are two examples of semi-characteristic functionals for $S$ since $\tilde{\phi}_S(x)$ and $\tilde{\phi}'_S(x) = 1$ if $x \in S$. Moreover, $\tilde{\phi}_S(x) = 1$ (or $\tilde{\phi}'_S(x) = 1$) if and only if

$$x_f = \tilde{x}_f = \tilde{x}(T) .$$

In the next section we shall see an application of these semi-characteristic functionals in the particular case $P = \overline{P}'$ given by (4.3.8). We shall assume that $\phi_S$ is a semi-characteristic functional which satisfies

$$\phi_S(x(\cdot),x_f) = 1 \implies x_f = \tilde{x}_f = \tilde{x}(T) \tag{4.6.4}$$

where $\tilde{x}(\cdot)$ and $\tilde{x}_f$ now denotes the components of $\overline{P}'x$ in $L^2(0,T;Z)$ and $Z$ respectively. That is

$$\overline{P}'(x(\cdot),x_f) = (\tilde{x}(\cdot),\tilde{x}_f) .$$

Clearly, by (4.1.10), any characteristic functional $\phi_S$ for $S$ satisfies (4.6.4). Also, if $P = \overline{P}'$, $\phi_S = \tilde{\phi}_S$ in (4.6.2) and $\phi_S = \tilde{\phi}'_S$ in (4.6.3) both satisfy (4.6.4).

4.6.2 - $\underline{S = R(S)}$.

Consider the set $S$ given by (4.2.4), namely

$$S = R(S) = \{(x_0, S(\cdot)x_0) \in X : x_0 \in Z\} .$$

We have seen in example 4.2.7 that $\phi_S$ given by (4.2.5) is a characteristic functional for $S$ . Many other characteristic functionals for $S$ could be easily found. For example, $\phi_S':X \to \mathbb{R}$

$$\phi_S'(x) = \frac{m\|x(\cdot) - S(\cdot)x_0\| + 1}{\|x(\cdot) - S(\cdot)x_0\| + 1} \quad , \quad m > 1 \qquad (4.6.5)$$

$\phi_S$ in (4.6.5) satisfies (4.6.1).

## 4.7 - UNIFORM PROJECTIONS.

In most of the applications of projections in chapters 6 and 7 we are free to choose any projection $P$ onto a given set, say $S_{ad}$ . However, in some cases we can obtain better results by using special type of projections which we study here.

In this section $X$ denotes a vector space of the type

$$X = X_1 \times X_2$$

where $X_1$ and $X_2$ are two vector spaces.

Let $x_1 \in X_1$ and $x_2 \in X_2$ , then $x = (x_1,x_2) \in X$ . We shall refer to $x_1$ and $x_2$ as, respectively, the first and the second component of $x$ .

### 4.7.1 - Uniform projections in the first component.

Here we introduce the concept of uniformity in the first component for projections $P:X \to X$ onto a subset $S$ of $X$ .

We say that $P$ is a uniform projection in the first component if it satisfies

$$P(x_1,x_2) = (x_1,\tilde{x}_2) \Rightarrow x_2 = \tilde{x}_2 \qquad (4.7.1)$$

that is, whenever the first component of $Px$ is the same as the first component of $x$ then the second component of $Px$ and $x$ are also the same.

Clearly if $P$ is uniform in the first component then we have that

$$P(x_1,x_2) = (x_1,\tilde{x}_2) \Rightarrow Px = x \quad \text{and} \quad x \in S \ .$$

We could also define, in a similar way, a uniform projection in the second component, however, the above definition is enough for our applications.

4.7.2 - Applications in State Estimation.

In chapter 7 we shall be interested in uniform projections in the first component onto $S$ given by (4.2.4) when

$$X = M^0(0,T;Z) = Z \times C(0,T;Z)$$

or

$$X = M^p(0,T;Z) = Z \times L^p(0,T;Z) \quad , \quad 1 \leq p < \infty \ .$$

We have seen in example 4.2.7 that $\bar{P}$ given by (4.2.3) and $\bar{P}'$ given by (4.2.6) are two continuous projections onto $S$ . One can easily check that $\bar{P}$ is not uniform in the first component but $\bar{P}'$ is. This is achieved by the characteristic functional $\phi_S$ in (4.2.5). In fact the result still holds if we replace $\phi_S$ by any other characteristic functional for $S$ . That is, for $\phi_S$ any characteristic functional onto $S$, $P':X \rightarrow X$

$$P'x = P'(x_0,x(\cdot)) = \phi_S(x)\cdot(x_0,Sx_0) \tag{4.7.2}$$

is a uniform projection in the first component onto $S$ .

4.7.3 - Applications in Control.

In our applications of control in chapter 6 we shall be interested in uniform projections in the first component onto the subspace $S = R(LB)$ when $X$ is a space of type $M(0,T;Z)$ (see §2.2.6) such as for example

$$X = M^s(0,T;Z) = H^s(0,T;Z) \times Z \qquad \text{for some} \quad s \in \mathbb{R} .$$

We have seen in example 4.3.6 that $\bar{P}$, $\bar{P}'$ and $\Pi$ given respectively by (4.3.7), (4.3.8) and (4.3.12) are three projections onto $S$ . Generally neither $\bar{P}$ nor $\bar{P}'$ is uniform in the first component. However, we shall see in the sequel that $\Pi$ is uniform in the first component and that we can also obtain such projections using semi-characteristic functionals for $S$ .

4.7.4 - Theorem.

If $R(L(T)B) = Z$ then we have that the projection $\Pi$ given by (4.3.12), namely $\Pi = \Pi_1 + \Pi_2$ ,

$$\Pi_1(x(\cdot),x_f) = LB(L(T)B)^{\dagger}x_f \quad , \qquad \Pi_2 = (I-\Pi_1)P$$

is a uniform projection in the first component onto $S$ , independent of the choice of $P$ .

Proof.

First note that, since $R(L(T)B) = Z$ , the following holds for all

$x(\cdot) \in H^s(0,T;Z)$

$$\Pi_1(x(\cdot),x_f) = \Pi_1(x(\cdot),\tilde{x}_f) \iff x_f = \tilde{x}_f \ . \qquad (4.7.3)$$

Also, for $u \in \mathcal{U}$

$$(I-\Pi_1)(L(\cdot)Bu,L(T)Bu) = (L(\cdot)Bu',0) \quad \text{for some} \quad u' \in \mathcal{U} \ .$$

Hence, since $Px \in S \Rightarrow Px = (L(\cdot)Bu,L(T)Bu)$ for some $u \in \mathcal{U}$ and therefore

$$\Pi_2 x = (I-\Pi_1)Px = (L(\cdot)Bu',0) \quad \text{for some} \quad u' \in \mathcal{U} \ . \qquad (4.7.4)$$

Equations (4.7.3) and (4.7.4) imply

$$\Pi(x(\cdot),x_f) = \Pi(x(\cdot),\tilde{x}_f) \iff x_f = \tilde{x}_f \ . \qquad (4.7.5)$$

Now suppose that

$$\Pi(x(\cdot),x_f) = (x(\cdot),\tilde{x}_f)$$

then, since $(x(\cdot)\tilde{x}_f) \in S$ , $(x(\cdot),\tilde{x}_f) = \Pi(x(\cdot),\tilde{x}_f)$ and therefore

$$\Pi(x(\cdot),x_f) = \Pi(x(\cdot),\tilde{x}_f)$$

and by (4.7.5) we have that $x_f = \tilde{x}_f$ . So $\Pi$ is uniform in the first component.

Q.E.D.

4.7.5 - A remark about $\Pi$ .

Consider the projection $\Pi_1$ and $\Pi_2$ (see example 4.3.6) and suppose that $\mathcal{R}(L(T)B) = Z$ . Note that

$$\Pi_1(z(\cdot),z_f) = (L(\cdot)Bu',z_f) \qquad \text{for some} \quad u' \in N(LB)^\perp$$

$$\Pi_2(z(\cdot),z_f) = (L(\cdot)Bu'',0) \qquad \text{for some} \quad u'' \in N(LB)^\perp \quad .$$

In other words, if $z = (z(\cdot),z_f) \in X$ , the second component of $\Pi_1 z$ is the same as the second component of $z$ and the second component of $\Pi_2 z$ is always $0$ . Also, $u'$ and $u''$ are such that

$$z_f = L(T)Bu' \qquad \text{and} \qquad L(T)Bu'' = 0 \ .$$

Clearly

$$\Pi(z(\cdot),z_f) = (L(\cdot)Bu,z_f) \qquad \text{for some} \quad u \in N(LB)^\perp$$

(namely $u = u' + u''$). That is, the second component of $z$ is also the second component of $\Pi z$ . In fact that is the reason why $\Pi$ is a (linear) uniform projection (as we have seen in theorem 4.7.4). We can also have nonlinear uniform projections by using semi-characteristic functionals for $S$ , as we shall see in the next two paragraphs:

4.7.6 - <u>Theorem</u>.

Let $X$, $S$ and $\bar{P}'$ be as in example 4.3.6 and $\phi_S$ be any semi-characteristic functional for $S$ which satisfies (4.6.4). (Note: this includes all characteristic-functionals for $S$ .) The mapping $\hat{P}:X \to X$ given by

$$\hat{P}x = \bar{P}'\left(\frac{x}{\phi_S(x)}\right) \tag{4.7.6}$$

is a continuous uniform projection in the first component onto $S$ .

Proof.

Clearly $\tilde{P}$ is a continuous projection (see example 4.1.8(ii)) onto $S = \mathcal{R}(LB)$ .

Now suppose that

$$\tilde{P}(x(\cdot),x_f) = (x(\cdot),\tilde{x}_f)$$

thus

$$\frac{L(\cdot)B(L(\cdot)B)^{\dagger}x(\cdot)}{\phi_S(x)} = x(\cdot)$$

which implies that $x(\cdot) \in \mathcal{R}(L(\cdot)B)$ , $x(\cdot) = L(\cdot)B(L(\cdot)B)^{\dagger}x(\cdot)$ and hence $\phi_S(x) = 1$ . Therefore, by (4.6.4), $x_f = \tilde{x}_f$ and this implies that $\tilde{P}$ is uniform in the first component.

Q.E.D.

4.7.7 - Theorem.

Let $X,S,\bar{P}'$ and $\phi_S$ as in lemma 4.7.5. The mapping $\tilde{P}':X \to X$ given by

$$\tilde{P}'x = \bar{P}'(\phi_S(x)\cdot x) \tag{4.7.7}$$

is a continuous uniform projection in the first component onto $S$ .

Proof.

Analogous to theorem 4.7.6.

□

4.7.8 - Uniform projections and bounded sets.

Unfortunately, if $S$ is a bounded subset of $X$ uniform projections are not immediate and sometimes they do not even exist. Consider the following example

$$S \text{ is a compact subset of } X = X_1 \times X_2$$

then, by theorem 4.1.4 any continuous projection $P$ onto $S$ is completely continuous. Since $S$ is bounded we can always find $\bar{x}_2 \in X_2$ such that

$$(x_1,\bar{x}_2) \notin S \quad \text{for all } x_1 \in X_1 . \tag{4.7.8}$$

Now we define $F: X \to X$ by

$$F(x_1,x_2) = P(x_1,\bar{x}_2)$$

and suppose that $P$ is also uniform in the first component. By Schauder's fixed point theorem (see section 2.6) $F$ has a fixed point since it is completely continuous. Set $(x_1^*,x_2^*)$ a fixed point of $F$, this implies that

$$(x_1^*,x_2^*) = P(x_1^*,\bar{x}_2) \in S$$

and hence, since we assumed $P$ to be uniform in the first component,

$$(x_1^*,x_2^*) = (x_1^*,\bar{x}_2) \in S$$

which is a contradiction to (4.7.8).

This example shows that if $S$ is compact then there is no continuous projection uniform in the first component onto $S$. We shall restrict our applications of uniform projections to the case $S$ is a subspace of $X$.

# CHAPTER 5.

## SOME ASPECTS OF LINEAR OPERATORS IN INNER-PRODUCT SPACES.

In this chapter we only deal with inner-product spaces. For an inner-product space, say $X$, sometimes we represent $<,>_X$ and $\|\cdot\|_X$ by just $<,>$ and $\|\cdot\|$ respectively. Similarly, for an operator $T \in \mathfrak{L}(U,X)$ we shall sometimes use $\|T\|$ for $\|T\|_{\mathfrak{L}(U,X)}$.

### 5.1 - INTRODUCTION.

An immediate corollary of the closed graph theorem (see §2.2.9) is that if the linear operator $T:U \to X$ is unbounded, then $\mathcal{D}(T) \neq U$, that is, $(\mathcal{D}(T), \|\cdot\|_U)$ is not a Hilbert space. We could however try to find another space $(U', \|\cdot\|_{U'})$ such that the same operator $T$ regarded now as being a transformation from $U'$ to $X$ is bounded and $\mathcal{D}(T) = U'$. We illustrate this with the following example: Let $U = X = L^2(0,1)$ and $T:U \to X$ be the differential operator $Tu = Du$, i.e.,

$$(Tu)(t) = \frac{du}{dt} \quad , \quad t \in [0,1] \,. \tag{5.1.1}$$

Clearly $T$ is unbounded (since it is not continuous) and $\mathcal{D}(T)$ which is the subspace

$$\mathcal{D}(T) = \{u \in L^2(0,1) : Tu \in L^2(0,1)\}$$

is dense in $U$. Now let $U'$ be the Sobolev space [1, 22]

$$U' = H^1(0,1)$$

which is the completion of $C^1(0,1)$ with respect to the norm $\|\cdot\|_{H^1}$

$$\|u\|_{H^1} = (\|u\|^2_{L^2} + \|Du\|^2_{L^2})^{\frac{1}{2}} \quad .$$

The differential operator $T$ in (5.1.1), regarded as being from $\mathcal{U}'$ to $\mathcal{X}$ is a bounded linear transformation, i.e., $T \in \mathcal{L}(\mathcal{U}',\mathcal{X})$, and $\mathcal{D}(T) = \mathcal{U}'$. Alternatively, we could interpret the differentiation (5.1.1) in the distributional sense [36] and in this case we can maintain $\mathcal{U} = L^2(0,1)$ and change $\mathcal{X}$ to $\mathcal{X}'$, the space of distributions

$$\mathcal{X}' = H^{-1}(0,1)$$

which is the dual space of $H^1_0(0,1)$ (see [22]). The operator $T:\mathcal{U} \to \mathcal{X}'$ defined by (5.1.1) is a bounded linear operator and $\mathcal{D}(T) = \mathcal{U}$.

The above example shows only two possibilities of how to adjust the spaces $\mathcal{U}$ and $\mathcal{X}$ to obtain a bounded $T$. We shall see in section 5.6 that there are an endless number of ways to choose $\mathcal{U}'$ and $\mathcal{X}'$ such that $T \in \mathcal{L}(\mathcal{U}',\mathcal{X}')$. We may however be interested in other topological properties of the operator $T$ such as, for example, if the range of $T:\mathcal{U} \to \mathcal{X}$ is a closed subspace of $\mathcal{X}$. Consider the following example: Let $\mathcal{U} = \mathcal{X} = L^2(0,1)$ and $T:\mathcal{U} \to \mathcal{X}$ be the integral operator given by

$$(Tu)(t) = \int_0^t u(s)ds \quad , \quad t \in [0,1] . \tag{5.1.2}$$

Although now $T$ is a bounded operator, $T$ does not have closed range, that is the set

$$R(T) = \{x(\cdot) : x(t) = \int_0^t u(s)ds \text{ for some } u(\cdot) \in L^2(0,1)\}$$

is not a closed set in $L^2(0,1)$ . We could think of finding some spaces of functions $\mathcal{U}'$ and $\mathcal{X}'$ such that the operator T defined by (5.1.2) will have closed range when regarded as a linear transformation from $\mathcal{U}'$ to $\mathcal{X}'$ . We shall see in section 5.6 that this is also possible and in fact there are an endless number of choices of such spaces $\mathcal{U}'$ and $\mathcal{X}'$ .

In this chapter we shall see that if $\mathcal{U}$ and $\mathcal{X}$ are two inner product spaces and $T:\mathcal{U} \to \mathcal{X}$ a linear operator, then we can adjust the spaces $\mathcal{U}$ and $\mathcal{X}$ obtaining $\mathcal{U}'$ and $\mathcal{X}'$ such that some topological properties of the operator T , for example

$$T \text{ is bounded,} \tag{5.1.3}$$

or

$$T \text{ has closed range,} \tag{5.1.4}$$

or

$$T \text{ is completely continuous,} \tag{5.1.5}$$

etc., holds for $T:\mathcal{U}' \to \mathcal{X}'$ .

As a matter of fact, in our applications, property (5.1.4) will be of even greater importance than (5.1.3) and we shall see that these two properties can hold independently of each other. If (5.1.5) holds then, obviously, (5.1.3) holds. However, we shall also see that (5.1.4) and (5.1.5) cannot hold simultaneously unless $\mathcal{R}(T)$ is finite dimensional.

The method we develop here enables us to choose $\mathcal{U}'$ and $\mathcal{X}'$ according to the topological properties of $T:\mathcal{U}' \to \mathcal{X}'$ we want and the flexibility of the problem to let $\mathcal{U}$ and/or $\mathcal{X}$ be altered.

If (5.1.4) holds for an operator $T:\mathcal{U}' \to \mathcal{X}'$ , that is, if

$R(T)$ is closed in $X'$

then, we have seen in §4.3.5 that

a) the generalized inverse $T^{\dagger} \in \mathcal{L}(X',U')$ (i.e. $T^{\dagger}$ is bounded).

Also, we shall see in section 5.2 that if $T$ is also bounded, then

b) $T$ maps open sets of $U'$ into open sets of $R(T)$ ; and

c) $T$ maps closed and bounded sets of $U'$ into closed and bounded sets of $X'$ .

Properties (a)-(c) will have applications in chapter 6 when we take $T = LB$ for $L$ as in (2.2.27) and $B$ as in example 2.2.5.

## 5.2 - BOUNDED OPERATORS AND OPERATORS OF CLOSED RANGE.

In this section we study some properties of bounded operators as well as operators of closed range. First we introduce a test to verify if $T$ has closed range:

### 5.2.1 - Theorem.

Let $T:U \to X$ be a linear operator, $U$ and $X$ Hilbert spaces. $R(T)$ is closed in $X$ if and only if there exists $\gamma_1 > 0$ such that

$$\|Tu\|_X \geq \gamma_1 \|u\|_U \quad \text{for all} \quad u \in N(T)^{\perp} . \tag{5.2.1}$$

Proof.

Let $\tilde{T}^{-1}:R(T) \to N(T)^{\perp}$ be given by (4.3.4). Clearly, since $(N(T)^{\perp},\|\cdot\|_U)$ is a Hilbert space and $\tilde{T}^{-1}$ is one-to-one, $R(T)$ is closed in $X$ if and only if

$$(R(T), \|\cdot\|_X) \text{ is a Hilbert space}$$

which is the case if and only if

$$\tilde{T}^{-1} \text{ is bounded}$$

which holds if and only if (5.2.1) holds.

Q.E.D.

We emphasize that equation (5.2.1) only holds for $u \in N(T)^{\perp}$ (e.g., take $u \neq 0$ , $u \in N(T)$ then $\|Tu\| = 0$ and $\|u\| > 0$ ). Also observe that T need not be a bounded operator to have closed range, that is (5.2.1) may hold independently of the existence of $\gamma_2 > 0$ such that

$$\|Tu\|_X \leq \gamma_2 \|u\|_U \quad \text{for all} \quad u \in U . \tag{5.2.2}$$

In other words, we may have T unbounded of closed range or also T bounded with range not closed, etc.

Let us now introduce a test to verify if T is bounded.

5.2.2 - <u>Theorem</u>.

Let $T:U \to X$ a linear operator, $U$ and $X$ Hilbert spaces. $T \in \mathcal{L}(U,X)$ if and only if there exists $\gamma_2 > 0$ such that

$$\|Tu\|_X \leq \gamma_2 \|u\|_U \quad \text{for all} \quad u \in N(T)^{\perp} . \tag{5.2.3}$$

Proof.

We only need to show that (5.2.3) => (5.2.2). Take $u \in U$ and set $u' = Pu$, where $P: U \to U$ is the orthogonal projection onto $N(T)^\perp$. Clearly (5.2.3) holds for $u'$, thus

$$\|TPu\|_X \leq \gamma_2 \|Pu\|_U \qquad \forall\, u \in U \ .$$

Now, using (i) of theorem 4.3.4 (i.e., $T = TP$) and (iii) of corollary 4.3.3 (i.e., $\|P\|_{\mathcal{L}(U)} = 1$), we have that

$$\|Tu\|_X = \|TPu\|_X \leq \gamma_2 \|Pu\|_U \leq \gamma_2 \|u\|_U \qquad \forall\, u \in U$$

and this concludes the proof.

Q.E.D.

5.2.3 - Corollary.

Let $U$ and $X$ be Hilbert spaces. $T \in \mathcal{L}(U,X)$ and $R(T)$ is closed if and only if there exists $\gamma_1, \gamma_2 > 0$ such that

$$\gamma_1 \|u\|_U \leq \|Tu\|_X \leq \gamma_2 \|u\|_U \quad \text{for all} \quad u \in N(T)^\perp \ . \qquad (5.2.4)$$

Proof.

Immediate from theorems 5.2.1 and 5.2.2.

Q.E.D.

Note that corollary 5.2.3 actually says that if $T$ is bounded and has closed range then $(N(T)^\perp, \|\cdot\|_U)$ and $(R(T), \|\cdot\|_X)$ are topologically isomorphic (i.e., $T/N(T)^\perp$ is an isomorphism and a homeomorphism simultaneously).

In section 2.2 we have seen that any continuous function maps compact sets into compact sets. Any linear operator maps convex sets into convex sets and any bounded linear operator maps bounded sets into bounded sets. We showed in theorems 4.4.1 and 4.4.4 that in Hilbert spaces, orthogonal projections map open sets into open sets of its range and closed and bounded sets into closed and bounded sets. We shall now see that this result extends for any bounded linear operator with closed range. (Also compare this result with equations (2.2.1) and (2.2.2) which say that for any continuous function, the inverse image of an open (respectively closed) set is open (respectively closed).)

5.2.4 - Theorem.

Let $\mathcal{U}$ and $\mathcal{X}$ be Hilbert spaces, $T \in \mathcal{L}(\mathcal{U},\mathcal{X})$ and $\mathcal{R}(T)$ be closed in $\mathcal{X}$. If $\mathcal{U}_{ad}$ is an open set in $\mathcal{U}$ then $T(\mathcal{U}_{ad})$ is open in $(\mathcal{R}(T),\|\cdot\|_{\mathcal{X}})$. If however $\mathcal{U}_{ad} \subset \mathcal{U}$ is a closed and bounded set then $T(\mathcal{U}_{ad})$ is also closed and bounded.

Proof.

Let $P$ be the orthogonal projection onto $\mathcal{N}(T)^{\perp}$, thus $P(\mathcal{U}_{ad}) \subset \mathcal{N}(T)^{\perp}$ for any $\mathcal{U}_{ad} \subset \mathcal{U}$. Using corollary 5.2.3, since $T = TP$ by (i) of theorem 4.3.4, $P(\mathcal{U}_{ad})$ is open (respectively closed) in $(\mathcal{N}(T)^{\perp},\|\cdot\|_{\mathcal{U}})$ if and only if $T(\mathcal{U}_{ad})$ is open (respectively closed) in $(\mathcal{R}(T),\|\cdot\|_{\mathcal{X}})$. Now, by theorems 4.4.1 and 4.4.4 the result follows.

Q.E.D.

Remark.

Since orthogonal projections have closed range (see section 4.3), theorems 4.4.1 and 4.4.4 become particular cases of the above theorem.

Now we present an almost immediate property of operators of closed range. Unlike some of the previous propositions, the following corollary has a generalization for Banach spaces.

5.2.5 - Corollary (of theorem 5.2.1).

Let $\mathcal{U}$ and $X$ be Hilbert spaces. If $T:\mathcal{U} \to X$ has closed range then $T$ is completely continuous if and only if $R(T)$ is finite dimensional.

Proof.

If $R(T)$ has finite dimension then $T$ is completely continuous (see §2.2.1). To see the converse when $R(T)$ is closed in $X$ let $\{e_n\}$ be a complete orthonormal set in $N(T)^{\perp}$. Then we can find a subsequence $\{e_n'\}$ of $\{e_n\}$ such that

$$\{Te_n'\} \text{ is linearly independent}$$

$$\overline{\text{Span}\{Te_n'\}} = \overline{R(T)} \ .$$

By (5.2.1), since $T$ has closed range, there is $\gamma_1 > 0$ such that

$$\|Te_n'\| \geq \gamma_1 \quad \text{for all } n \ . \tag{5.2.5}$$

Clearly (5.2.5) $\Rightarrow \{Te_n'\}$ has only a finite number of elements when $T$ is completely continuous, otherwise $\{Te_n'\}$ would be a sequence in $R(T)$ with no convergent subsequence.

Q.E.D.

The above result says that when $R(T)$ is infinite dimensional we cannot have $T$ completely continuous and $R(T)$ closed at the same time.

## 5.3 - THE GENERALIZED GRAM-SCHMIDT PROCESS.

We have seen in §2.4.6 that if we have a set of linearly independent vectors $E = \{\phi_n\}$, the Gram-Schmidt orthogonalization process generates an orthonormal set $E' = \{\phi_n'\}$ with the property that for each $k$,

$$\mathrm{Span}\{\phi_1,\ldots,\phi_k\} = \mathrm{Span}\{\phi_1',\ldots,\phi_k'\} .$$

Unfortunately if $\{\phi_n\}$ is not a linearly independent set this process does not work. Here we present a generalization of this process which allows cases where $\{\phi_n\}$ in any countable (finite or infinite) set of elements in $X$. We show for the case $\{\phi_n\}_{n=1,2,\ldots}$ is any sequence in $X$. The particularization for the case $\{\phi_n\}$ is a finite set will be obvious.

### 5.3.1 - Theorem (The generalized Gram-Schmidt process):

Let $E = \{\phi_n\}_{n=1,2,\ldots}$ be a countable set in $X$. Define the sequence $\{\psi_i\}_{i=1,2,\ldots}$ by

$$\psi_k = \begin{cases} 0 & \text{if } \tilde{\phi}_k = 0 \\ \dfrac{\tilde{\phi}_k}{\|\tilde{\phi}_k\|} & \text{if } \tilde{\phi}_k \neq 0 \end{cases}$$

where $\|\cdot\|$ is the norm in $X$ and $\{\tilde{\phi}_i\}_{i=1,2,\ldots}$ is given by

$$\tilde{\phi}_1 = \phi_1$$

and

$$\tilde{\phi}_k = \phi_k - \sum_{i=1}^{k-1} \langle\phi_k,\psi_i\rangle\psi_i , \qquad \text{if } k > 1 . \tag{5.3.1}$$

If $E' = \{\phi_n'\}_{n=1,2,\ldots}$ is the subsequence of $\{\psi_i\}_{i=1,2,\ldots}$ obtained by eliminating all $\psi_i$ such that $\psi_i = 0$, then $E'$ is an orthonormal set. Moreover, if $n_i : \mathbb{N} \to \mathbb{N}$ is the mapping that sets $n \to n_i$ where $n_i$ is the original position of $\phi_n'$ in the sequence $\{\psi_i\}_{i=1,2,\ldots}$ then, for each $n = 1,2,\ldots$ we have

$$\mathrm{Span}\{\phi_1',\ldots,\phi_n'\} = \mathrm{Span}\{\phi_1,\ldots,\phi_{n_i}\} \quad .$$

Proof: First we see that for any $i \geq 1$

$$\|\psi_i\| = 1 \qquad \text{if} \quad \psi_i \neq 0 \; . \tag{5.3.2}$$

Also note that $\psi_1$ satisfies

$$\mathrm{Span}\{\psi_1\} = \mathrm{Span}\{\phi_1\} \quad .$$

Now let $P_i : X \to X$ be the orthogonal projection onto $E_i = \mathrm{Span}\{\psi_1,\ldots,\psi_{i-1}\}$ and observe that the second part of the RHS of (5.3.1) is the action of $P_i$ on $\phi_i$. Therefore

$$\tilde{\phi}_i = \phi_i - P_i\phi_i = (I-P_i)\phi_i \qquad \text{for any} \quad i > 1 \; .$$

Thus, since $(I-P_i)$ is the orthogonal projection onto $E_i^{\perp}$, $\tilde{\phi}_i \in E_i^{\perp}$ and this implies that for any $i > 1$

$$\psi_i = \tilde{\phi}_i = 0 \qquad \Longleftrightarrow \qquad \phi_i \in E_i \; .$$

With the above results one can easily obtain, by induction, that for a generic $k \geq 1$

$$\mathrm{Span}\{\psi_1,\ldots,\psi_k\} = \mathrm{Span}\{\phi_1,\ldots,\phi_k\} \tag{5.3.3}$$

and

$$\psi_j \perp \psi_\ell \qquad \text{for} \quad j \neq \ell \; , \qquad 1 \leq j,\ell \leq k \; . \tag{5.3.4}$$

By (5.3.2) and (5.3.4) the subsequence $\{\phi'_n\}_{n=1,2,\ldots}$ , obtained from $\{\psi_i\}_{i=1,2,\ldots}$ by taking only $\psi_i \neq 0$ , is orthonormal. By (5.3.3) we have

$$\mathrm{Span}\{\phi'_1,\ldots,\phi'_n\} = \mathrm{Span}\{\psi_1,\ldots,\psi_{n_i}\} = \mathrm{Span}\{\phi_1,\ldots,\phi_{n_i}\}$$

and this completes the proof. Q.E.D.

## 5.4 - COMPLETION OF AN ORTHONORMAL SET.

In functional analysis "completeness" is widely used with two different meanings:

a) complete spaces [ 3 ], being spaces in which every Cauchy sequence converges. (This definition is valid in a general context of metric spaces.)

b) complete orthonormal sets [ 3 ]: An orthonormal set E in an inner-product space $X$ is said to be complete if there is no other orthonormal set $E' \neq E$ in $X$ such that $E' \supset E$ . (This definition is only valid in the context of inner-product spaces.)

The concept of "completion" however, has always been used in the sense of definition (a) (see p.120 of [32]). In fact every metric space has a unique completion (see p.121 of [32]). In this section we shall define "completion" with respect to orthonormal sets, that is, in the sense of definition (b). Again, as in the case of completeness, the two concepts of completion will be distinct from each other. We remark that completion as defined here will not be unique.

5.4.1 - Definition of completion for orthonormal sets:

We have seen in §2.4.1 that if $E' = \{\phi'_n\}$ is an orthonormal set in $X$ then there exists another set $E''$ such that

$$E' \cup E'' \text{ is a complete orthonormal set in } X .$$

We shall call $E''$ a completion for $E'$ in $X$ .

Observe that if $E'$ is a complete orthonormal set in $X$ then $E'' = \emptyset$ , the empty set, is the only completion for $E'$ . In §5.4.4 we give a generalization of the above definition for any linearly independent set. We shall now present a procedure which generates $E''$ , a completion of $E'$ in $X$ , once $E'$ and a complete orthonormal set $\{\overline{\phi}_n\}$ in $X$ is given.

5.4.2 - Theorem:

Let $\{\overline{\phi}_n\}_{n=1,2,...}$ be a complete orthonormal set in $X$ and $E' = \{\phi'_n\}_{n=1,2,...}$ an orthonormal set in $X$ . Define the sequence $\{\psi_i\}_{i=1,2,...}$ by

$$\psi_i = \begin{cases} 0 & \text{if } \tilde{\phi}_i = 0 \\ \dfrac{\tilde{\phi}_i}{\|\tilde{\phi}_i\|} & \text{if } \tilde{\phi}_i \neq 0 \end{cases}$$

where $\{\tilde{\phi}_i\}_{i=1,2,...}$ is given by

$$\tilde{\phi}_1 = \bar{\phi}_1 - \sum_n <\bar{\phi}_1,\phi_n'>\phi_n'$$

$$\tilde{\phi}_i = \bar{\phi}_i - \sum_n <\bar{\phi}_i,\phi_n'>\phi_n' - \sum_{j=1}^{i-1} <\bar{\phi}_i,\psi_j>\psi_j \,, \qquad \text{if} \quad i > 1 \,.$$

If $E" = \{\phi_n"\}_{n=1,2,...}$ is the subsequence of $\{\psi_i\}_{i=1,2,...}$ obtained by eliminating all elements $\psi_i$ such that $\psi_i = 0$ , then

$$\{\phi_n'\}_{n=1,2,...} \cup \{\phi_n"\}_{n=1,2,...} \text{ is a complete orthonormal set in } X \,.$$

(i.e., E" is a completion for E' in $X$ .)

<u>Proof</u>: Set $X' = \overline{\text{Span}(E')}$ and $X" = (X')^{\perp}$ , then $X = X' \oplus X"$ .

Let $P':X \to X$ and $P":X \to X$ be the orthogonal projections onto $X'$ and $X"$ respectively. Clearly, since $E' = \{\phi_n'\}_{n=1,2,...}$ is a complete orthonormal set in $X'$ ,

$$P'\bar{\phi}_i = \sum_n <\bar{\phi}_i,\phi_n'>\phi_n' \,.$$

Thus, $\{\tilde{\phi}_i\}_{i=1,2,...}$ can be expressed by

$$\tilde{\phi}_1 = (I-P')\bar{\phi}_1 \tag{5.4.1}$$

$$\tilde{\phi}_i = (I-P')\bar{\phi}_i - \sum_{j=1}^{i-1} <\phi_i,\psi_j>\psi_j \,, \qquad \text{if} \quad i > 1 \,. \tag{5.4.2}$$

From (5.4.1), since $(I-P') = P"$ , $\psi_1 \in X"$ and, using property 4.1.2,

$$\psi_1 = \tilde{\phi}_1 = 0 \qquad \Longleftrightarrow \qquad \bar{\phi}_1 \in X' \,.$$

This implies that

$$\mathrm{Span}(E' \cup \{\psi_1\}) = \mathrm{Span}(E' \cup \{\overline{\phi}_1\})$$

and

$$\psi_1 \perp \phi'_n \qquad \text{for } n = 1,2,\ldots \ .$$

Now let $P_i : X \to X$ be the orthogonal projection onto $E_i = \mathrm{Span}\{\psi_1,\ldots,\psi_{i-1}\}$, then the last part of the RHS of (5.4.2) represents the action of $P_i$ on $\overline{\phi}_i$, that is

$$\tilde{\phi}_i = [I-(P' + P_i)]\overline{\phi}_i \qquad \text{for } i > 1 \ . \tag{5.4.3}$$

Clearly $E_1 \perp X'$. One can easily show, by induction, that $E_i \perp X'$ for each $i = 1,2,\ldots$ . This implies that $(P' + P_i)$ is the orthogonal projection onto $X' \oplus E_i$ and hence $I-(P'+P_i)$ is the orthogonal projection onto $(X' \oplus E_i)^\perp$ . Now, using (5.4.3), one can easily verify that $\psi_i \in X''$ and

$$\psi_i = \tilde{\phi}_i = 0 \qquad \Longleftrightarrow \qquad \overline{\phi}_i \in X' \oplus E_i \qquad \text{for } i > 1 \ .$$

Therefore we have for any $k \geq 1$

$$\mathrm{Span}(E' \cup \{\psi_1,\ldots,\psi_k\}) = \mathrm{Span}(E' \cup \{\overline{\phi}_1,\ldots,\overline{\phi}_k\}) \tag{5.4.4}$$

and

$$\psi_j \perp \psi_\ell \qquad \text{for } 1 \leq j, \ell \leq k \ . \tag{5.4.5}$$

By (5.4.5), since $\|\psi_i\| = 1$ if $\psi_i \neq 0$, the subsequence

$E'' = \{\phi''_n\}_{n=1,2,\ldots}$ , generated from $\{\psi_n\}$ be eliminating all $\psi_i$ such that $\psi_i = 0$ , is orthonormal. By (5.4.4) we obtain

$$\mathrm{Span}(E' \cup E'') = \mathrm{Span}(E' \cup \{\overline{\phi}_n\}_{n=1,2,\ldots}) .$$

Clearly this implies that $E' \cup E'' = \{\phi'_n\}_{n=1,2,\ldots} \cup \{\phi''_n\}_{n=1,2,\ldots}$ is complete since $\{\overline{\phi}_n\}_{n=1,2,\ldots}$ is complete. Q.E.D.

5.4.3 - Remarks:

i) Theorem 5.4.2 presents a procedure for finding a completion $E''$ for any orthonormal set $E'$ in $X$ . It is important to observe that if $E' = \{\phi'_n\}$ is not orthonormal then theorem 5.4.2 will not hold. This follows since in this case $P'$ will not be an orthogonal projection and therefore $P' + P_i$ in (5.4.3) will not be (in general) a projection at all.

ii) If $E' = \{\phi'_n\}$ is a complete orthonormal set then the sequence $\{\psi_i\}$ generated by the above process will be

$$\{\psi_i\} = \{0,0,\ldots\}$$

and therefore, as one would expect, $E'' = \emptyset$ , the empty set.

iii) If $X''$ is a finite dimensional space, then any completion for $E'$ in $X$ will have a finite number of elements. The particularization of the process for this case is obvious. □

5.4.4 - Completion for linearly independent sets:

Suppose we have $E' = \{e'_n\}$ a linearly independent set in $X$ and we want to find $E'' = \{e''_n\}$, another set of linearly independent vectors in $X$ such that

$$\overline{\mathrm{Span}(\{e'_n\} \cup \{e''_n\})} = \overline{X} .$$

Again in this sense we shall call $E''$ a completion for $E'$ in $X$.

If $E'$ is not orthonormal, it is necessary to obtain the orthonormalization of $E'$, say $\{\hat{e}_n\}$, by using the Gram-Schmidt process and then apply the method of theorem 5.4.2 to obtain $E'' = \{e''_n\}$, a completion for $\{\hat{e}_n\}$ in $X$. Clearly, since

$$\overline{\mathrm{Span}\{e'_n\}} = \overline{\mathrm{Span}\{\hat{e}_n\}} ,$$

such $E''$ will also be a completion for $E'$ in $X$ in the above sense. Moreover,

$$\overline{\mathrm{Span}\{e'_n\}} \perp \overline{\mathrm{Span}\{e''_n\}} . \tag{5.4.6}$$

5.5 - PRIMITIVE OPERATORS AND COMPLETE MATCHED SETS.

Let $T:\mathcal{U} \to X$ be a linear operator densely defined, that is

$$\mathcal{D}(T) = \mathcal{U} \quad \text{or} \quad \mathcal{D}(T) \quad \text{is dense in} \quad \mathcal{U} .$$

Here we shall abandon the topology given by $<,>_{\mathcal{U}}$ and $<,>_X$ and define vector subspaces $[U]$ and $[X]$ of $U$ and $X$ respectively, which will be called primitive subspaces of $U$ and $X$. Then we consider the action of T algebraically between $[U]$ and $[X]$. In section 5.6 we introduce topologies to $[U]$ and $[X]$ in order to obtain new spaces $(U',<,>_{U'})$ and $(X',<,>_{X'})$ which will be Hilbert spaces and the operator T regarded as a transformation from $U'$ to $X'$ will have the desired topological properties.

5.5.1 - Primitive Subspaces:

If $\{e_n\}$ and $\{\phi_n\}$ are two complete orthonormal sets in $U$ and $X$ respectively, then clearly $U \supset \text{Span}\{e_n\}$ and $X \supset \text{Span}\{\phi_n\}$ and if $U$ and $X$ are Hilbert spaces then

$$U = \overline{\text{Span}\{e_n\}} \qquad \text{and} \qquad X = \overline{\text{Span}\{\phi_n\}} \ ,$$

where the bars over $\text{Span}\{e_n\}$ and $\text{Span}\{\phi_n\}$ denote the completion of these vector spaces in the topology induced by $<,>_U$ and $<,>_X$ respectively. In particular, if $U$ is finite dimensional then $U = \text{Span}\{e_n\}$ and similarly if $X$ is finite dimensional, $X = \text{Span}\{\phi_n\}$.

We define $[U]$ to be a primitive subspace of $U$ if $[U]$ is the vector space defined by

$$[U] = \text{Span}\{e_n\}$$

for some complete orthonormal set $\{e_n\}$ in $U$. In other words, a primitive subspace $[U]$ of $U$ is the set of all linear combinations of a finite number of elements of a complete orthonormal set in $\mathcal{U}$ .

It is easy to verify that if $\mathcal{U}$ is finite dimensional, then $\mathcal{U} = [\mathcal{U}] = \mathrm{Span}\{e_n\}$ , independent on the choice of the complete orthonormal set $\{e_n\}$ . However, if $\mathcal{U}$ is infinite dimensional then $[\mathcal{U}]$ is not unique since it depends on the choice of the complete orthonormal set $\{e_n\}$ . As a matter of fact $([\mathcal{U}],<,>_{\mathcal{U}})$ is the smallest dense subspace of $\mathcal{U}$ which contains $\{e_n\}$ . This follows since every subspace of $\mathcal{U}$ which contains $\{e_n\}$ must also contain $\mathrm{Span}\{e_n\}$ .

Suppose that $\Gamma$ represents the set of indices of $\{e_n\}$ . For a typical element $u \in [\mathcal{U}]$ we can always find a finite set $J \subset \Gamma$ such that $u$ can be expressed in the form

$$u = \sum_{n \in J} u_n e_n \tag{5.5.1}$$

for some scalars $u_n \in \mathbb{F}$ . That is, $u$ is the linear combination of a finite number of elements of $\{e_n\}$ .

Similarly we can define a primitive subspace of $X$ by

$$[X] = \mathrm{Span}\{\phi_n\}$$

for some complete orthonormal set $\{\phi_n\}$ in $X$ .

5.5.2 - Primitive operators:

We shall call a primitive operator any linear transformation $T^1$ from a primitive subspace $[\mathcal{U}]$ of $\mathcal{U}$ to $X$ . In particular, if

$$T^1 u = Tu \qquad\qquad \text{for all } \; u \in [\mathcal{U}]$$

then we say that $T^1$ is a primitive operator of $T$ .

Observe that a primitive operator $T^1$ is only defined in $[\mathcal{U}]$ . This implies that if $T^1$ is the primitive operator of $T$ then $T^1$ , regarded as a transformation from $\mathcal{U}$ to $\mathcal{X}$ , is densely defined. However, we shall not look at $T^1$ as a transformation between the two normed vector spaces $\mathcal{U}$ and $\mathcal{X}$ but, instead (as in the above definition) as a linear transformation between the vector spaces $[\mathcal{U}]$ and $\mathcal{X}$ . So there is no sense in considering continuity of $T^1$ or if the range of $T^1$ is closed, or any other topological property for $T^1$ .

If $T^1 : [\mathcal{U}] \to \mathcal{X}$ is a primitive operator of $T$ then, for a typical element $u \in [\mathcal{U}]$ given by (5.5.1), $T^1 u$ has the form

$$T^1 u = \sum_{n \in J} u_n (Te_n) \ .$$

It is clear that the set $T^1([\mathcal{U}])$ is contained in some primitive subspace $[\mathcal{X}]$ of $\mathcal{X}$ .

5.5.3 - Matched sets:

Let $T^1 : [\mathcal{U}] \to \mathcal{X}$ be a primitive operator and the sets $\Delta, \Gamma$ and $\Lambda$ be three countable sets satisfying

$\Delta \subseteq \Gamma \cap \Lambda$

$\Delta$ is a nonempty set.

Also let $\{e_n\}_{n\in\Gamma}$ be a sequence of linearly independent vectors such that

$$e_n \in [U] \qquad \text{for all } n \in \Gamma$$

and $\{\phi_n\}_{n\in\Lambda}$ be a sequence of linearly independent vectors such that for some primitive subspace $[X]$ of $X$,

$$\phi_n \in [X] \qquad \text{for all } n \in \Lambda \ .$$

We say that the quintuple

$$M = (\{e_n\}_{n\in\Gamma}\ ,\ \{\phi_n\}_{n\in\Lambda}\ ,\ \Delta,\Gamma,\Lambda)$$

is a matched set for $T^1$ if

$$T^1 e_n = \phi_n \qquad \text{for all } n \in \Delta$$

$$T^1 e_n = 0 \qquad \text{for all } n \in \Gamma\backslash\Delta \ .$$

We call a matched set $M$ complete if

$$\mathrm{Span}\{e_n\}_{n\in\Gamma} = [U] \qquad \text{and} \qquad \mathrm{Span}\{\phi_n\}_{n\in\Lambda} = [X] \ .$$

We also use the notation

$$\Gamma' = \Gamma\backslash\Delta \qquad \text{and} \qquad \Lambda' = \Lambda\backslash\Delta$$

so that we have

$$\Gamma = \Delta \cup \Gamma' \qquad \text{and} \qquad \Delta \cap \Gamma' = \emptyset \ ,$$

$$\Lambda = \Delta \cup \Lambda' \qquad \text{and} \qquad \Delta \cap \Lambda' = \emptyset \ .$$

Observe that we allow the possibility of $\Gamma'$ or $\Lambda'$ (or both) being empty sets.

5.5.4 - Examples of complete matched sets:

i) Let $T:\mathcal{U} \to \mathcal{X}$ be the differential operator given by (5.1.1), namely

$$(Tu)(t) = \frac{du}{dt}$$

where $\mathcal{U} = \mathcal{X} = L^2(0,1)$ over $\mathbb{R}$. Also let $E$ and $E'$ be

$$E = \{1\} \cup \{\sqrt{2} \cos n\pi t\}_{n\in \mathbb{N}} \quad \text{and} \quad E' = \{\sqrt{2} \sin n\pi t\}_{n\cup \mathbb{N}} .$$

Setting $[\mathcal{U}] = \mathrm{Span}(E)$ and $[\mathcal{X}] = \mathrm{Span}(E')$, since $E$ is a complete orthonormal set in $\mathcal{U}$ and $E'$ is a complete orthonormal set in $\mathcal{X}$, we obtain primitive subspaces $[\mathcal{U}]$ and $[\mathcal{X}]$ of $\mathcal{U}$ and $\mathcal{X}$ respectively. Clearly $[\mathcal{U}]$ and $[\mathcal{X}]$ can be expressed as

$$[\mathcal{U}] = \mathrm{Span}\{\sin n\pi t\}_{n\in \mathbb{N}} \quad \text{and}$$

$$[\mathcal{X}] = \mathrm{Span}(\{n\pi \cos n\pi t\}_{n\in \mathbb{N}} \cup \{1\}) .$$

An element $u \in [\mathcal{U}]$ can be expressed in the form (5.5.1) by

$$u = \sum_{n\in J} u_n \sin n\pi t \tag{5.5.2}$$

for some $J \subset \mathbb{N}$, $J$ a finite set, dependent on $u$. Now define $T^1:[\mathcal{U}] \to \mathcal{X}$ by

$$T^1 u = \sum_{n\in J} u_n \, n\pi \cos n\pi t \tag{5.5.3}$$

for any $u$ given by (5.5.2). Clearly $T^1$ is a primitive operator of $T$.

Now take $\Gamma = \Delta = \mathbb{N} = \{1,2,\ldots\}$ , $\Lambda = \{0,1,2,\ldots\}$ . Thus $\Gamma' = \emptyset$ and $\Lambda' = \{0\}$. Let the sets $\{e_n\}_{n\in\Gamma}$ , $\{\phi_n\}_{n\in\Lambda}$ be defined by

$$e_n(t) = \sin n\pi t \qquad \text{for all } n \in \Gamma = \Delta$$

$$\phi_n(t) = n\pi \cos n\pi t \qquad \text{for all } n \in \Delta \quad \text{and}$$

$$\phi_0(t) = 1 \ .$$

Since $Te_n = \phi_n$ for all $n \in \Delta$ , $\text{Span}\{e_n\}_{n\in\Gamma} = [U]$ , $\text{Span}\{\phi_n\}_{n\in\Lambda} = [X]$ and both $\{e_n\}_{n\in\Gamma}$ and $\{\phi_n\}_{n\in\Lambda}$ are linearly independent sets, it is clear that $M = (\{e_n\}_{n\in\Gamma} , \{\phi_n\}_{n\in\Lambda} ,\Delta,\Gamma,\Lambda)$ is a complete matched set for the primitive operator $T^1$ given by (5.5.3).

ii) Let $T:U \to X$ be the integral operator given by (5.1.2), namely

$$(Tu)(t) = \int_0^t u(s)ds$$

where $U = X = L^2(0,1)$ over $\mathbb{R}$ . We have seen in section 5.1 that $T$ is densely defined in $U$ . Set

$$E = \{1\} \cup \{\sqrt{2} \sin 2n\pi t\}_{n\in \mathbb{N}} \cup \{\sqrt{2} \cos 2n\pi t\}_{n\in \mathbb{N}}$$

and $[U] = [X] = \text{Span}(E)$ . Since $E$ is a complete orthonormal set in $L^2(0,1)$ , $[U]$ and $[X]$ are primitive subspaces of $U$ and $X$ respectively. Clearly we can write

$$[U] = \text{Span}(\{2n\pi \cos 2n\pi t\}_{n\in \mathbb{N}} \cup \{-2n\pi \sin 2n\pi t\}_{n \in \mathbb{N}} \cup \{1\})$$

$$[X] = \text{Span}(\{\sin 2n\pi t\}_{n\in\mathbb{N}} \cup \{\cos 2n\pi t\}_{n\in \mathbb{N}} \cup \{1\}) \quad .$$

An element $u \in [U]$ can be expressed in the form (5.5.1) by

$$u = u_0 + \sum_{n\in J} 2n\pi(u_n' \cos 2n\pi t - u_n'' \sin 2n\pi t) \tag{5.5.4}$$

where $u_0 \in \mathbb{R}$ , $u_n'$ , $u_n'' \in \mathbb{R}$ for all $n \in J$ and $J \subset \mathbb{N}$ is a set with a finite number of elements, dependent on $u$ . If we define $T^1 : [X] \to Y$ by

$$T^1 u = u_0 t + \sum_{n\in J} (u_n' \sin 2\pi t + u_n'' \cos 2n\pi t) \tag{5.5.5}$$

for any $u$ given by (5.5.4), then $T^1$ is clearly a primitive operator of $T$ . Now set

$$\Gamma = \Lambda = \Delta = \{0, \pm 1, \pm 2, \dots\}$$

so that $\Gamma' = \Lambda' = \emptyset$ (the empty set). Define the sequences $\{e_n\}_{n\in\Gamma}$ and $\{\phi_n\}_{n\in\Lambda}$ by

$$e_n(t) = \begin{cases} 1 & \text{if } n = 0 \\ 2n\pi \cos 2n\pi t & \text{if } n = 1,2,\dots \\ 2n\pi \sin 2|n|\pi t & \text{if } n = -1,-2,\dots \end{cases}$$

$$\phi_n(t) = \begin{cases} t & \text{if } n = 0 \\ \sin 2n\pi t & \text{if } n = 1,2,\dots \\ \cos 2|n|\pi t & \text{if } n = -1,-2,\dots \end{cases}$$

Clearly we have $Te_n = \phi_n$ for all $n \in \Delta$ and both $\{e_n\}_{n\in\Gamma}$ and $\{\phi_n\}_{n\in\Lambda}$ are linearly independent. Furthermore $\text{Span}\{e_n\}_{n\in\Gamma} = [U]$ and $\text{Span}\{\phi_n\}_{n\in\Lambda} = [X]$ . So $M = (\{e_n\}_{n\in\Gamma}, \{\phi_n\}_{n\in\Lambda}, \Delta, \Lambda, \Gamma)$ is a complete matched set for the primitive operator $T^1$ defined by (5.5.5). □

5.5.5 - <u>Remark:</u> Clearly matched sets are not unique. For example, if

$$M = (\{e_n\}_{n\in\Gamma} \ , \ \{\phi_n\}_{n\in\Lambda} \ ,\Delta,\Lambda,\Gamma)$$

is a (complete) matched set for some primitive operator $T^1$ then, for any three scalars $k, k_1, k_2 \in \mathbb{F}$ ,

$$M = (\{\tilde{e}_n\}_{n\in\Gamma},\{\tilde{\phi}_n\}_{n\in\Lambda},\Delta,\Lambda,\Gamma)$$

where

$$\tilde{e}_n = \begin{cases} k\ e_n & , \quad n\in\Delta \\ \\ k_1 e_n & , \quad n\in\Gamma' = \Gamma\backslash\Delta \end{cases} \qquad \text{and} \qquad \tilde{\phi}_n = \begin{cases} k\ \phi_n & , \quad n\in\Delta \\ \\ k_2\phi_n & , \quad n\in\Lambda' = \Lambda\backslash\Delta \end{cases}$$

is also a (complete) matched set for $T^1$ . □

## 5.6 - THE SPACES $\mathcal{U}'$ AND $\mathcal{X}'$

Here $\mathcal{U}$ and $\mathcal{X}$ are again separable inner-product spaces over $\mathbb{F}$ (which is either $\mathbb{R}$ or $\mathbb{C}$ ), $[\mathcal{U}]$ and $[\mathcal{X}]$ are primitive spaces of $\mathcal{U}$ and $\mathcal{X}$ respectively, $T^1:[\mathcal{U}] \to \mathcal{X}$ is a primitive operator and

$$M = (\{e_n\}_{n\in\Gamma} \ , \ \{\phi_n\}_{n\in\Lambda} \ ,\Delta,\Gamma,\Lambda)$$

is a complete matched set for $T^1$ .

5.6.1 - Definition of the spaces $\mathcal{U}' = \mathcal{U}'(e_n,\alpha_n,\Gamma)$ and $\mathcal{X}' = \mathcal{X}'(\phi_n,\beta_n,\Lambda)$:

Let $\alpha_n$ and $\beta_n$ be real numbers satisfying

$$\alpha_n > 0 \qquad\qquad \text{for all} \quad n \in \Gamma$$

$$\beta_n > 0 \qquad\qquad \text{for all} \quad n \in \Lambda \ .$$

We define the normed vector spaces $\mathcal{U}' = \mathcal{U}'(e_n, \alpha_n, \Gamma)$ by

$$\mathcal{U}' = \{u = \sum_{n\in\Gamma} u_n e_n : \quad u_n \in \mathbb{F} \quad \text{and} \quad (\sum_{n\in\Gamma} \alpha_n |u_n|^2) < \infty\}$$

$$\| u \|_{\mathcal{U}'} = (\sum_{n\in\Gamma} \alpha_n |u_n|^2)^{\frac{1}{2}}$$

and $\mathcal{X}' = \mathcal{X}'(\phi_n, \beta_n, \Lambda)$ by

$$\mathcal{X}' = \{x = \sum_{n\in\Lambda} x_n \phi_n : \quad x_n \in \mathbb{F} \quad \text{and} \quad (\sum_{n\in\Lambda} \beta_n |x_n|^2) < \infty\}$$

$$\| x \|_{\mathcal{X}'} = (\sum_{n\in\Lambda} \beta_n |x_n|^2)^{\frac{1}{2}} .$$

Let $u, v \in \mathcal{U}'$ , $u = \sum_{n\in\Gamma} u_n e_n$ , $v = \sum_{n\in\Gamma} v_n e_n$ and define the inner-product in $\mathcal{U}'$ by

$$\langle u,v\rangle_{\mathcal{U}'} = \sum_{n\in\Gamma} \alpha_n u_n \overline{v}_n \quad .$$

Similarly, for $x, y \in \mathcal{X}'$ , $x = \sum_{n\in\Lambda} x_n \phi_n$ , $y = \sum_{n\in\Lambda} y_n \phi_n$ , define the inner product in $\mathcal{X}'$ by

$$\langle x,y\rangle_{\mathcal{X}'} = \sum_{n\in\Lambda} \beta_n x_n \overline{y}_n \quad .$$

It is easy to verify that $\| \cdot \|_{\mathcal{U}'}$ is the norm induced by $\langle , \rangle_{\mathcal{U}'}$ and similarly $\| \cdot \|_{\mathcal{X}'}$ is the norm induced by $\langle , \rangle_{\mathcal{X}'}$ , the spaces $\mathcal{U}'$ and $\mathcal{X}'$ are Hilbert spaces and the sets

$$E_{\mathcal{U}'} = \left\{ \frac{e_n}{\sqrt{\alpha_n}} \right\}_{n\in\Gamma} \qquad \text{and} \qquad E_{\mathcal{X}'} = \left\{ \frac{\phi_n}{\sqrt{\beta_n}} \right\}_{n\in\Lambda}$$

are complete orthonormal sets in $\mathcal{U}'$ and $\mathcal{X}'$ respectively. Also,

$$\mathcal{U}' = \overline{[\mathcal{U}]} = \overline{\text{Span}(E_{\mathcal{U}'})}$$

and

$$\mathcal{X}' = \overline{[\mathcal{X}]} = \overline{\text{Span}(E_{\mathcal{X}'})}$$

where now the bars denote the completion of $[\mathcal{U}]$ and $[\mathcal{X}]$ in the topology given by $\|\cdot\|_{\mathcal{U}'}$ and $\|\cdot\|_{\mathcal{X}'}$ respectively.

Observe that for each $\{\alpha_n\}_{n\in\Gamma}$ we have a different space $\mathcal{U}'$, that is, $\mathcal{U}'$ is parametrized by the numbers $\alpha_n$. Similarly, $\mathcal{X}'$ is parametrized by the numbers $\beta_n$.

5.6.2 - <u>The operator T</u> :

We shall now define a linear operator $T:\mathcal{U}' \to \mathcal{X}'$. First suppose that we have started our problem with an original operator $\tilde{T}:\mathcal{U} \to \mathcal{X}$ and $T^1:[\mathcal{U}] \to \mathcal{X}$ is a primitive operator of $\tilde{T}$. We shall see that $T:\mathcal{U}' \to \mathcal{X}'$ defined here will be the same operator $\tilde{T}:\mathcal{U} \to \mathcal{X}$ but regarded as being a transformation from $\mathcal{U}'$ to $\mathcal{X}'$.

Define the linear operator $T:\mathcal{U}' \to \mathcal{X}'$ by

$$Tu = \sum_{n\in\Delta} u_n\phi_n \tag{5.6.1}$$

for $u \in \mathcal{D}(T)$, where $\mathcal{D}(T)$ is defined by

$$\mathcal{D}(T) = \{u = \sum_{n\in\Gamma} u_n e_n \in \mathcal{U}' : \quad (\sum_{n\in\Delta} \beta_n |u_n|^2) < \infty\} . \tag{5.6.2}$$

Since $\mathcal{D}(T) \supset [\mathcal{U}] = \text{Span}(E_{\mathcal{U}'})$, $T$ is densely defined in $\mathcal{U}'$. Moreover, it is easy to check that $T$ is a closed operator and

$$Tu = T^1 u \qquad \text{for all} \quad u \in [U] \ .$$

Now suppose that $T^1$ is a primitive operator of some linear operator from $U$ to $X$, say $\tilde{T}: U \to X$, then

$$Tu = \tilde{T}u \qquad \text{for all} \quad u \in [U]$$

that is, $T$ and $\tilde{T}$ coincide in a dense subset of their domains. In other words, $T$ is the operator $\tilde{T}$ regarded as a transformation from $U'$ to $X'$.

One can verify that $\overline{N(T)} = U'(e_n, \alpha_n, \Gamma')$ and $N(T)^\perp = U'(e_n, \alpha_n, \Delta)$, i.e.,

$$\overline{N(T)} = \{u = \sum_{n\in\Gamma'} u_n e_n : \quad (\sum_{n\in\Gamma'} \alpha_n |u_n|^2) < \infty\} \tag{5.6.3}$$

$$N(T)^\perp = \{u = \sum_{n\in\Delta} u_n e_n : \quad (\sum_{n\in\Delta} \alpha_n |u_n|^2) < \infty\} \tag{5.6.4}$$

and $\overline{R(T)} = X'(\phi_n, \beta_n, \Delta)$ and $R(T)^\perp = X'(\phi_n, \beta_n, \Lambda')$, i.e.,

$$\overline{R(T)} = \{x = \sum_{n\in\Delta} x_n \phi_n : \quad (\sum_{n\in\Delta} \beta_n |x_n|^2) < \infty\} \tag{5.6.5}$$

$$R(T)^\perp = \{x = \sum_{n\in\Lambda'} x_n \phi_n : \quad (\sum_{n\in\Lambda'} \beta_n |x_n|^2) < \infty\}. \tag{5.6.6}$$

Clearly we have that

$$\overline{N(T)} \oplus N(T)^\perp = U' \qquad \text{and} \qquad \overline{R(T)} \oplus R(T)^\perp = X' \ .$$

5.6.3 - <u>The adjoint operator of T</u> :

It is not difficult to verify that the adjoint of $T$ is the operator

$T^*:\mathcal{D}(T^*) \to \mathcal{U}'$ given by

$$T^*x = \sum_{n\in\Delta} \frac{\beta_n}{\alpha_n} x_n e_n \tag{5.6.7}$$

for $x \in \mathcal{D}(T^*)$ , where $\mathcal{D}(T^*)$ is defined by

$$\mathcal{D}(T^*) = \{ x = \sum_{n\in\Lambda} x_n\phi_n : \quad (\sum_{n\in\Delta} \beta_n|x_n|^2) < \infty\} \supset \mathcal{X}' .$$

<u>Note</u> that $\overline{\mathcal{R}(T^*)} = \mathcal{N}(T)^\perp$ and $\overline{\mathcal{R}(T)} = \mathcal{N}(T^*)^\perp$ (c.f. theorem 4.3.4).

5.6.4 - <u>The generalized inverse of T</u> :

Again, one can easily check that the generalized inverse of T (see 4.3.5) is given by $T^\dagger:\mathcal{D}(T^\dagger) \to \mathcal{U}'$ defined by

$$T^\dagger x = \sum_{n\in\Delta} x_n e_n \tag{5.6.8}$$

for $x \in \mathcal{D}(T^\dagger)$ , where

$$\mathcal{D}(T^\dagger) = \{x = \sum_{n\in\Lambda} x_n\phi_n : \quad (\sum_{n\in\Delta} \alpha_n|x_n|^2) < \infty\} .$$

5.6.5 - <u>Projections:</u>

The orthogonal projections $P:\mathcal{U}' \to \mathcal{U}'$ and $P_1:\mathcal{U}' \to \mathcal{U}'$ onto respectively $\mathcal{N}(T)^\perp$ and $\overline{\mathcal{N}(T)}$ are given by

$$Pu = \sum_{n\in\Delta} u_n e_n \quad \text{and} \quad P_1 u = (I-P)u = \sum_{n\in\Gamma'} u_n e_n \tag{5.6.9}$$

for $u = \sum_{n\in\Gamma} u_n e_n \in \mathcal{U}'$ . The orthogonal projections $\bar{P}:\mathcal{X}' \to \mathcal{X}'$ and $\bar{P}_1:\mathcal{X}' \to \mathcal{X}'$ onto respectively $\overline{\mathcal{R}(T)}$ and $\mathcal{R}(T)^{\perp}$ are given by

$$\bar{P}x = \sum_{n\in\Delta} x_n\phi_n \qquad \text{and} \qquad \bar{P}_1 x = (I-\bar{P})x = \sum_{n\in\Lambda'} x_n\phi_n \tag{5.6.10}$$

for $x = \sum_{n\in\Lambda} x_n\phi_n \in \mathcal{X}'$ .

5.6.6 - <u>The conjugate spaces of $\mathcal{U}'$ and $\mathcal{X}'$</u> :

Let $\mathcal{U}^*$ denote the conjugate space of $\mathcal{U}' = \mathcal{U}'(e_n,\alpha_n,\Gamma)$ , i.e., $\mathcal{U}^* = \mathcal{L}(\mathcal{U}',\mathbb{R})$ , the space consisting of all bounded linear functionals defined on $\mathcal{U}'$ . Similarly we denote $\mathcal{X}^*$ the conjugate space of $\mathcal{X}' = \mathcal{X}'(\phi_n,\beta_n,\Lambda)$ . We have that

$$\mathcal{U}^* = \mathcal{U}'(e_n, \frac{1}{\alpha_n}, \Gamma)$$

and

$$\mathcal{X}^* = \mathcal{X}'(\phi_n, \frac{1}{\beta_n}, \Lambda) .$$

Clearly $(\mathcal{U}^*)^* = \mathcal{U}'$ and $(\mathcal{X}^*)^* = \mathcal{X}'$ , as one would expect, since $\mathcal{U}'$ and $\mathcal{X}'$ are Hilbert space and therefore reflexive.

## 5.7 - <u>THREE THEOREMS ON COMPLETE MATCHED SETS</u>.

Here we consider $T:\mathcal{U}' \to \mathcal{X}'$ , where the spaces $\mathcal{U}' = \mathcal{U}'(e_n,\alpha_n,\Gamma)$

and $X' = X'(\phi_n,\beta_n,\Lambda)$ (dependent on the choice of $\alpha_n$'s and $\beta_n$'s) were defined in section 5.6.1 and the densely defined operator T was defined in section 5.6.2. We shall establish here some results which will allow us to choose the numbers $\alpha_n$ and $\beta_n$ such that T, regarded as a transformation from $U'$ to $X'$, has some topological properties such as: T is a bounded operator or T has closed range, etc.

5.7.1 - <u>First theorem</u>:

$T \in \mathcal{L}(U',X')$ if and only if there exists a real number $M > 0$ such that

$$\frac{\beta_n}{\alpha_n} \leq M \qquad \text{for all} \quad n \in \Delta . \tag{5.7.1}$$

<u>Proof</u>: First we see that for $u \in N(T)^{\perp}$ (given by (5.6.4)), we have

$$\| Tu \|^2_{X'} = \sum_{n\in\Delta} \beta_n |u_n|^2 \qquad \text{and} \qquad \| u \|^2_{U'} = \sum_{n\in\Delta} \alpha_n |u_n|^2 . \tag{5.7.2}$$

Thus, for $u \in N(T)^{\perp}$ one can verify that (5.7.1) holds if and only if there exists $M > 0$ such that

$$\| Tx \|_{X'} \leq \sqrt{M} \| x \|_{U'}$$

and therefore, by theorem 5.2.2, the result follows. Q.E.D.

5.7.2 - <u>Corollary</u>:

$T \in \mathcal{L}(U',X')$ if and only if the set $\{\beta_n/\alpha_n\}_{n\in\Delta}$ is bounded.

<u>Proof</u>: Immediate by (5.7.1) since $\frac{\beta_n}{\alpha_n} > 0$ for all $n \in \Delta$ . Q.E.D.

<u>Remark</u>: Note that if $T \in \mathcal{L}(\mathcal{U}',\mathcal{X}')$ then one can easily verify that $\mathcal{D}(T) = \mathcal{U}'$ , $\mathcal{N}(T) = \overline{\mathcal{N}(T)}$ and $T^* \in \mathcal{L}(\mathcal{X}',\mathcal{U}')$ , which are well known results in functional analysis. Moreover,

$$\| T \|_{\mathcal{L}(\mathcal{U}',\mathcal{X}')} = \| T^* \|_{\mathcal{L}(\mathcal{X}',\mathcal{U}')} = \sup_{n\in\Delta} \{\sqrt{\frac{\beta_n}{\alpha_n}}\} .$$

5.7.3 - <u>Second theorem</u>:

$T:\mathcal{U}' \to \mathcal{X}'$ has closed range if and only if there exists a real number $m > 0$ such that

$$\frac{\beta_n}{\alpha_n} \geq m \qquad \text{for all } n \in \Delta . \tag{5.7.3}$$

<u>Proof</u>: Since for $u \in \mathcal{N}(T)^{\perp}$ $\| Tu \|_{\mathcal{X}'}$ and $\| u \|_{\mathcal{U}'}$ are given by (5.7.2), one can verify that (5.7.3) holds if and only if there exists $m > 0$ such that

$$\| Tu \|_{\mathcal{X}'} \geq \sqrt{m} \| u \|_{\mathcal{U}'} .$$

Hence, using theorem 5.2.1, the result follows. Q.E.D.

5.7.4 - <u>Corollary</u>:

$T:\mathcal{U}' \to \mathcal{X}'$ has closed range if and only if the set $\{\alpha_n/\beta_n\}_{n\in\Delta}$ is bounded.

Proof: Clearly (5.7.3) holds if and only if the set $\{\alpha_n/\beta_n\}_{n\in\Delta}$ is bounded from above with an upper bound $1/m$. Since $(\alpha_n/\beta_n) > 0$ for all $n \in \Delta$ then $\{\alpha_n/\beta_n\}_{n\in\Delta}$ is also bounded from below and the result follows. Q.E.D.

Remark: If $T: U' \to X'$ has closed range then, as one would expect, $R(T) = \overline{R(T)}$, the generalized inverse $T^\dagger$ is bounded and

$$\| T^\dagger \|_{\mathcal{L}(U',X')} = \sup_{n\in\Delta} \{\sqrt{\frac{\alpha_n}{\beta_n}}\} . \qquad \square$$

5.7.5 - Corollary:

$T \in \mathcal{L}(U',X')$ and $R(T)$ is closed in $X'$ if and only if there exist real numbers $M$, $m > 0$ such that

$$m \le \frac{\beta_n}{\alpha_n} \le M \qquad \text{for all } n \in \Delta . \tag{5.7.4}$$

Moreover, (5.7.4) holds if and only if the sets $\{\alpha_n/\beta_n\}_{n\in\Delta}$ and $\{\beta_n/\alpha_n\}_{n\in\Delta}$ are both bounded.

Proof: Immediate from theorems 5.7.1 and 5.7.3 and corollaries 5.7.2 and 5.7.4. Q.E.D.

Remark: If $T \in \mathcal{L}(U',X')$ and has closed range then both $T$ and $T^\dagger$ are continuous, the orthogonal projection $P: U' \to U'$ onto $N(T)^\perp$, given by (5.6.9) can be expressed by (c.f. section 4.3.5)

$$P = T^\dagger T$$

and the orthogonal projection $\bar{P}:X' \to X'$ onto $\mathcal{R}(T)$ given by (5.6.10) can be expressed by (c.f. section 4.3.5)

$$\bar{P} = TT^{\dagger} \quad . \qquad \square$$

5.7.6 - Third theorem:

$T:\mathcal{U}' \to X'$ is a Hilbert-Schmidt operator (and therefore completely continuous) if and only if

$$\sum_{n\in\Delta} \frac{\beta_n}{\alpha_n} < \infty. \tag{5.7.5}$$

Proof: Since $E_{\mathcal{U}'} = \{e_n/\sqrt{\alpha_n}\}_{n\in\Gamma}$ is a complete orthonormal set in $\mathcal{U}'$ and

$$T\left(\frac{e_n}{\sqrt{\alpha_n}}\right) = \begin{cases} \dfrac{\phi_n}{\sqrt{\alpha_n}} & \text{if } n \in \Delta \\ 0 & \text{if } n \in \Gamma' = \Gamma\backslash\Delta \end{cases}$$

we have that

$$\sum_{n\in\Gamma} \left\| T\left(\frac{e_n}{\sqrt{\alpha_n}}\right) \right\|^2 = \sum_{n\in\Delta} \frac{\beta_n}{\alpha_n} \tag{5.7.6}$$

which implies that $T$ is a Hilbert-Schmidt operator (see §2.4.11) if and only if (5.7.5) holds. Q.E.D.

Remark: Since (5.7.5) => (5.7.1), by theorems 5.7.1 and 5.7.6 this is equivalent to say that

$T$ is Hilbert-Schmidt operator => $T$ is bounded

which is a well known result (see §2.4.11). However, (5.7.5) and (5.7.3) hold together if and only if $\mathcal{R}(T)$ has finite dimension. By theorems 5.7.3 and 5.7.6 this implies that a Hilbert-Schmidt operator $T$ has closed range if and only if $\mathcal{R}(T)$ is a finite dimensional space. Indeed, we have seen this result in corollary 5.2.5. □

5.7.7 - Corollary:

Let $\mathcal{U}' = \mathcal{U}'(e_n,\alpha_n,\Gamma)$ and $\mathcal{X}' = \mathcal{X}'(\phi_n,\beta_n,\Lambda)$ be as defined in §5.6.1. Suppose that $\Gamma = \Lambda$ and for some $c_n \in \mathbb{R}$, $\phi_n = c_n e_n$ for all $n \in \Gamma$, then we have the following result: $\mathcal{U}' = \mathcal{X}'$ and $\|\cdot\|_{\mathcal{U}'}$ is equivalent to $\|\cdot\|_{\mathcal{X}'}$ if and only if both the sets

$$\left\{ \frac{\alpha_n c_n^2}{\beta_n} \right\}_{n\in\Gamma} \quad \text{and} \quad \left\{ \frac{\beta_n}{\alpha_n c_n^2} \right\}_{n\in\Gamma} \quad \text{are bounded.} \tag{5.7.7}$$

(In the sequel we shall use the notation $\mathcal{U}' \simeq \mathcal{X}'$ if $\mathcal{U}' = \mathcal{X}'$ and $\|\cdot\|_{\mathcal{U}'}$ is equivalent to $\|\cdot\|_{\mathcal{X}'}$.) In other words this corollary says that if (5.7.7) holds $\mathcal{U}'$ and $\mathcal{X}'$ have the same elements, however the topology in $\mathcal{U}'$ is such that $\{e_n/\sqrt{\alpha_n}\}_{n\in\Gamma}$ is a complete orthonormal set whereas the topology in $\mathcal{X}'$ is such that $\{c_n e_n/\sqrt{\beta_n}\}_{n\in\Gamma}$ is a complete orthonormal set.

Proof: Consider the identity operator $I:\mathcal{U}' \to \mathcal{X}'$. Clearly, by corollary 5.2.3, if we have $I$ bounded with range closed in $\mathcal{X}'$ then the spaces $\mathcal{U} = N(I)^{\perp}$ and $\mathcal{X}' = R(I)$ will have equivalent topologies. Take $u \in \mathcal{U}' \Rightarrow u = \sum_{n\in\Gamma} u_n e_n$, $\| u \|^2_{\mathcal{U}'} = \sum_{n\in\Gamma} |u_n|^2 \alpha_n$. Hence

$Iu = \sum_{n\in\Gamma} \frac{u_n}{c_n} \cdot c_n e_n = \sum_{n\in\Gamma} \frac{u_n}{c_n} \phi_n$ and therefore

$$\frac{\| Iu \|^2_{\mathcal{X}'}}{\| u \|^2_{\mathcal{U}'}} = \frac{\sum_{n\in\Gamma} |u_n|^2 \frac{\beta_n}{c_n^2}}{\sum_{n\in\Gamma} |u_n|^2 \alpha_n} \tag{5.7.8}$$

By corollary 5.2.3 $I:\mathcal{U}' \to \mathcal{X}'$ is bounded and has closed range if and only if

$$\gamma_1 \le \frac{\| Iu \|_{\mathcal{X}'}}{\| u \|_{\mathcal{U}'}} \le \gamma_2 \qquad \text{for some } \gamma_1, \gamma_2 > 0$$

and by (5.7.8) this will be the case if and only if

$$m \le \frac{\beta_n}{\alpha_n c_n^2} \le M \qquad \text{for some } M, m > 0$$

which is the same as (5.7.7). This conclude the proof. Q.E.D.

5.7.8 - Corollary (IMBEDDINGS):

Let $\mathcal{X}' = \mathcal{X}'(\phi_n, \beta_n, \Lambda)$ be as defined in §5.6.1 and $\mathcal{H}$ be a Hilbert space such that $\phi_n \in \mathcal{H}$ for all $n \in \Lambda$. If the set

$$\left\{ \frac{\| \phi_n \|_H^2}{\beta_n} \right\}_{n\in\Lambda} \text{ is bounded} \tag{5.7.9}$$

then $X'$ is continuously imbedded in $H$ .

In particular, if $\overline{\text{Span}\{\phi_n\}} = H$ and both sets

$$\left\{ \frac{\beta_n}{\| \phi_n \|_H^2} \right\}_{n\in\Lambda} \quad \text{and} \quad \left\{ \frac{\| \phi_n \|_H^2}{\beta_n} \right\}_{n\in\Lambda} \text{ are bounded} \tag{5.7.10}$$

then there exists a topological isomorphism $I:X' \to H$ . That is, there exists a linear bijective map I from $X'$ to $H$ such that for some $\gamma_1,\gamma_2 > 0$,

$$\gamma_1 \| x \|_{X'} \leq \| Ix \|_H \leq \gamma_2 \| x \|_{X'} \qquad \text{for all} \quad x \in X' .$$

In other words, if both sets $\{\beta_n/\| \phi_n \|_H^2\}_{n\in\Lambda}$ and $\{\| \phi_n \|_H^2/\beta_n\}_{n\in\Lambda}$ are bounded and $\overline{\text{Span}\{\phi_n\}} = H$ then $X'$ can be interpreted as being $H$ with a topology in which $\phi_n/\sqrt{\beta_n}$ are the orthogonal elements with unitary norm.

Remark: We shall also denote this topological isomorphism by

$$X' \simeq H \quad .$$

Also, we shall often set

$$\beta_n = \| \phi_n \|_H^2$$

so that (5.7.10) will hold and I will be an isometry.

Proof: First we see that if (5.7.9) holds then, for some $m > 0$ ,

$$\beta_n \geq m \| \phi_n \|_H^2 \qquad \text{for all} \quad n \in \Lambda \quad .$$

Take $x \in X' \Rightarrow x = \sum_{n\in\Lambda} x_n\phi_n$ , $\| x \|_{X'}^2 = \sum_{n\in\Lambda} |x_n|^2\beta_n$ . Then,

$$\| x \|_H^2 = \| \sum_{n\in\Lambda} x_n\phi_n \|_H^2 \leq \sum_{n\in\Lambda} |x_n|^2 \| \phi_n \|_H^2 \leq \frac{1}{m} \sum_{n\in\Lambda} |x_n|^2\beta_n = \| x \|_{X'}^2 \quad .$$

So, $\| x \|_H \le \| x \|_{X'}$ and hence the imbedding $X' \to H$ is continuous.
Now suppose that (5.7.10) holds. Then, for some $m, M > 0$

$$m \| \phi_n \|_X^2 \le \beta_n \le M \| \phi_n \|_X^2 \qquad \text{for all} \quad n \in \Lambda . \tag{5.7.11}$$

Also suppose that $\overline{\mathrm{Span}\{\phi_n\}_{n\in\Lambda}} = H$ . Then we can take a c.o.s.

$$\{\psi_n\}_{n\in\Lambda} \quad \text{in} \quad H$$

(with the same countable set of indices $\Lambda$). Let $I : X' \to H$ be

$$Ix = \sum_{n\in\Lambda} x_n \psi_n \qquad \text{for all} \quad x = \sum_{n\in\Lambda} x_n \phi_n \in X' \quad .$$

Then, using (5.7.11),

$$\frac{1}{M} = \frac{\sum_{n\in\Lambda} |x_n|^2 \| \psi_n \|^2}{\sum_{n\in\Lambda} |x_n|^2 . \| \phi_n \|_H^2 . M} \le \frac{\| Ix \|_H^2}{\| x \|_{X'}^2} \le \frac{\sum_{n\in\Lambda} |x_n|^2 \| \psi_n \|_H^2}{\sum_{n\in\Lambda} |x_n|^2 \| \phi_n \|_H^2 m} = \frac{1}{m}$$

and therefore

$$\frac{1}{\sqrt{M}} \| x \|_{X'} \le \| Ix \|_H \le \frac{1}{\sqrt{m}} \| x \|_{X'} \quad .$$

This conclues the proof.

Q.E.D.

5.7.9 - Examples:

i) Let $U = U'(e_n, \alpha_n, \Gamma)$ where

$$e_n(t) = \cos \frac{n\pi}{T} t \, , \quad \alpha_n = 1 \quad \text{and} \quad \Gamma = \mathbb{N} = \{1,2,\ldots\} \quad .$$

By the imbedding corollary 5.7.8, since $\| e_n(\cdot) \|^2_{L^2(0,T)} = \frac{1}{2}$ for all

$n = 1,2,\ldots$ , we have that $\mathcal{U}$ has an equivalent topology to $L^2(0,T)$ , however the following

$$\mathcal{U} \simeq L^2(0,T)$$

does not hold because $\mathrm{Span}\{e_n\} \neq \mathrm{Span}\{L^2(0,T)\}$ . In fact

$$\mathrm{Span}(\{e_n\} \cup \{1\}) = \mathrm{Span}\{L^2(0,T)\} \quad .$$

So $\mathcal{U}$ here can be regarded as a closed subspace in $L^2(0,T)$ or as the subspace of $L^2(0,T)$ without the one-dimensional subspace of the constant functions. Also, $\{\cos(n\pi/T)t\}_{n=1,2,\ldots}$ form a complete orthonormal set in $\mathcal{U}$ whereas $\{\sqrt{2}\cos(n\pi/T)t\}_{n=1,2,\ldots} \cup \{1\}$ form a complete orthonormal set in $L^2(0,T)$ .

ii) Let $\mathcal{U} = \mathcal{U}'(e_n,\alpha_n,\Gamma)$ where

$$e_n(t) = t^n \quad , \qquad \alpha_n = \frac{T^{(2n+1)}}{(2n+1)} \qquad \text{and} \qquad \Gamma = \mathbb{Z}^+ = \{0,1,2,\ldots\} \; .$$

By the imbedding corollary 5.7.8, since $\| e_n(\cdot) \|^2_{L^2(0,T)} = \alpha_n$ , we have that $\mathcal{U} \simeq L^2(0,T)$ . In other words, here $\mathcal{U}$ is the $L^2(0,T)$ functions with a topology in which the functions $1,t,t^2,\ldots$ are orthogonal. (Note that $e_n(T) = T^n$ for all $n = 0,1,2,\ldots$ .)

iii) Let $\mathcal{U}$ be as in the previous example and $\tilde{\mathcal{U}} = \mathcal{U}'(\tilde{e}_n,\tilde{\alpha}_n,\Gamma)$ where

$$\tilde{e}_n(t) = \frac{t^n}{T^n} \quad , \qquad \tilde{\alpha}_n = \frac{T}{(2n+1)} \qquad \text{and} \qquad \Gamma = \mathbb{Z}^+ = \{0,1,2,\ldots\} \quad .$$

Clearly $\tilde{e}_n = c_n e_n$ for $c_n = 1/T^n$ and $e_n$ as in example (ii) above. Since $\alpha_n c_n^2/\tilde{\alpha}_n = 1$ for all $n = 0,1,2,\ldots$, we have that, by corollary 5.7.7, $\tilde{U} \simeq U \simeq L^2(0,T)$. The only difference is that in $\tilde{U}$ the functions $1, \frac{t}{T}, \frac{t^2}{T^2}, \ldots$ are orthogonal. (Note that $\tilde{e}(T) = 1$ for all $n = 0,1,2,\ldots$ .) We could also have deduced that $\tilde{U} \simeq L^2(0,T)$ by applying the imbedding corollary 5.7.8 directly.

iv) Let $T$ be the differential operator given by (5.1.1), namely

$$(Tu)(t) = \frac{du}{dt}$$

where now we shall regard $T$ as being a transformation between the spaces $U' = U'(e_n,\alpha_n,\Gamma)$ and $X' = X'(\phi_n,\beta_n,\Lambda)$ defined in §5.6.1 using the complete matched set $M = (\{e_n\}_{n\in\Gamma},\{\phi_n\}_{n\in\Lambda},\Delta,\Gamma,\Lambda)$ obtained in example 5.5.4(i). If we set

$$\alpha_n = \alpha = \text{constant} \qquad \text{for} \qquad n = 1,2,\ldots$$

$$\beta_0 = \tfrac{1}{2} \qquad \text{and} \qquad \beta_n = n^2\pi^2 \qquad \text{for} \qquad n = 1,2,\ldots$$

then $\alpha_n/\|e_n\|^2_{L^2(0,1)} = 2\alpha$ for all $n \in \Gamma$ and $\beta_n/\|\phi_n\|^2_{L^2(0,1)} = 2$ for all $n \in \Lambda$. By corollary 5.7.8 we have that $U' \simeq L^2(0,1)$ and $X' \simeq L^2(0,1)$ too. Now, since $\{\beta_n/\alpha_n\}_{n\in\Delta} = \{n^2\pi^2\}_{n=1,2,\ldots}$ is not a bounded set we have that (applying corollary 5.7.2), $T \notin \mathcal{L}(U',X')$. In fact we already expected this result (see section 5.1). Now suppose we set

$$\alpha_n = \alpha = \text{constant} \qquad \text{for} \qquad n = 1,2,\ldots$$

$$\beta_n = \beta = \text{constant} \qquad \text{for} \qquad n = 0,1,2,\ldots$$

then, by corollary 5.7.8, we have that $\mathcal{U}' \simeq L^2(0,1)$ and $\mathcal{X}' \simeq H^{-1}(0,1)$ (since $\beta_n / \| \phi_n \|^2_{H^{-1}(0,1)} = 2\beta$ for all $n \in \Lambda$) . By corollary 5.7.5, since both sets $\{\alpha_n/\beta_n\}_{n\in\Delta}$ and $\{\beta_n/\alpha_n\}_{n\in\Delta}$ are bounded, we have that

$$T \in \mathcal{L}(\mathcal{U}',\mathcal{X}') \qquad \text{and} \qquad \mathcal{R}(T) \text{ is closed in } \mathcal{X}'. \tag{5.7.12}$$

Also, $\| T \|_{\mathcal{L}(\mathcal{U}',\mathcal{X}')} = \sqrt{\beta/\alpha}$ and $\| T^{\dagger} \|_{\mathcal{L}(\mathcal{X}',\mathcal{U}')} = \sqrt{\alpha/\beta}$ . We obtained (5.7.12) by setting suitable values for $\beta_n$ . The space $\mathcal{X}' = \mathcal{X}'(\phi_n,\beta_n,\Lambda)$ is larger than the original space $L^2(0,1)$ . We could also have obtained (5.7.12) by maintaining $\mathcal{X}' = L^2(0,1)$ and setting suitable values for $\alpha_n$ . For example, let

$$\alpha_n = n^2\pi^2 \qquad \text{for} \qquad n = 1,2,\ldots$$

$$\beta_0 = \tfrac{1}{2} \qquad \text{and} \qquad \beta_n = n^2\pi^2 \qquad \text{for} \qquad n = 1,2,\ldots \; .$$

then $\alpha_n / \| e_n \|^2_{H^1(0,1)} = 2n^2\pi^2 \,/\, (1+n^2\pi^2)$ and hence both $\{\alpha_n / \| e_n \|^2_{H^1(0,1)}\}_{n\in\Delta}$ and $\{\| e_n \|^2_{H^1(0,1)}/\alpha_n\}_{n\in\Delta}$ are bounded and therefore, by corollary 5.7.8, $\mathcal{U}' \simeq H^1(0,1)$ . Also, $\mathcal{X}' \simeq L^2(0,1)$ . Since $\alpha_n/\beta_n = 1$ for $n \in \Delta$ then, by corollary 5.7.5, we have that (5.7.12) holds here too. Moreover, now $\| T \|_{\mathcal{L}(\mathcal{U}',\mathcal{X}')} = \| T^{\dagger} \|_{\mathcal{L}(\mathcal{X}',\mathcal{U}')} = 1$.

If however we want

$$T \text{ is completely continuous} \tag{5.7.13}$$

then we could set, for example

$$\alpha_n = n^2\pi^2 , \qquad n = 1,2,\ldots \qquad \text{and} \qquad \beta_n = \beta = \text{constant}, \quad n = 0,1,2,\ldots$$

in which case $U' \simeq H^1(0,1)$ and $X' \simeq H^{-1}(0,1)$ . By theorem 5.7.6, since $\sum_{n\in\Delta} (\beta_n/\alpha_n) < \infty$ , we have that (5.7.13) holds. Alternatively, setting

$$\alpha_n = \alpha = \text{constant} \qquad \text{for} \quad n = 1,2,\ldots$$

$$\beta_0 = 1 \text{ and } \beta_n = \frac{1}{n^2\pi^2} \qquad \text{for} \quad n = 1,2,\ldots$$

we have $U' \simeq L^2(0,1)$ , $X' \simeq H^{-2}(0,1)$ and again (5.7.13) holds.

v) Let $T$ be the integral operator given by (5.1.2), namely

$$(Tu)(t) = \int_0^t u(s)ds$$

where now we regard it as a transformation between the spaces $U' = U'(e_n,\alpha_n,\Gamma)$ and $X' = X'(\phi_n,\beta_n,\Lambda)$ defined in §5.6.1 using the complete matched set $M = (\{e_n\}_{n\in\Gamma},\{\phi_n\}_{n\in\Lambda},\Delta,\Gamma,\Lambda)$ obtained in example 5.5.4(ii). In this case if we set

$$\alpha_0 = 1 \qquad \text{and} \qquad \alpha_n = 4n^2\pi^2 \qquad \text{for} \quad n = 1,2,\ldots$$

$$\beta_n = \beta = \text{constant} \qquad \text{for} \quad n = 0,1,2,\ldots$$

then $U' \simeq L^2(0,1)$ and $X' \simeq L^2(0,1)$ . Since $(\alpha_n/\beta_n) = (4n^2\pi^2)/\beta$ we have

$$\left\{\frac{\alpha_n}{\beta_n}\right\}_{n\in\Delta} \quad \text{is not bounded}$$

and therefore, by corollary 5.7.4, $T$ does not have closed range. However, $(\beta_n/\alpha_n) = \beta/(4n^2\pi^2)$ and hence

$$\{\frac{\beta_n}{\alpha_n}\}_{n\in\Delta} \quad \text{is bounded}$$

which implies, by corollary 5.7.2, that $T$ is bounded. In fact we already expected this result (see section 5.1). Also observe that $\sum_{n\in\Delta} (\beta_n/\alpha_n) < \infty$, thus (using theorem 5.7.6)

$T$ is completely continuous.

Now suppose we set

$$\alpha_n = \beta_n = 1 = \text{constant} \qquad \text{for all } n = 0,1,2,...$$

then (using the imbedding corollary 5.7.8) we now have

$$\mathcal{U}' \simeq H^{-1}(0,1) \qquad \text{and} \qquad \mathcal{X}' \simeq L^2(0,1)\ .$$

Clearly, by corollary 5.7.5, since

$$(\alpha_n/\beta_n) = (\beta_n/\alpha_n) = \beta = \text{constant} \qquad \text{for } n \in \Delta \tag{5.7.14}$$

we have that

$$T:\mathcal{U}' \to \mathcal{X}' \text{ is bounded} \qquad \text{and} \qquad \text{has closed range.} \tag{5.7.15}$$

Alternatively, setting

$$\alpha_0 = \beta_0 = 1 \qquad \text{and} \qquad \alpha_n = \beta_n = 4n^2\pi^2 \qquad \text{for } n = 1,2,...$$

we have (by applying the imbedding corollary 5.7.8 again)

$$\mathcal{U}' \simeq L^2(0,1) \qquad \text{and} \qquad \mathcal{X}' \simeq H^1(0,1)\ .$$

Here again we have that (5.7.14) holds and hence (5.7.15) holds too.

## 5.8 - THE GENERATION OF COMPLETE MATCHED SETS.

In this section use the abbreviation "c.o.s." for complete orthonormal sets. Suppose we have a linear operator $T:\mathcal{U} \to \mathcal{X}$, densely defined, and a c.o.s. $E = \{\bar{\phi}_n\}$ in $\mathcal{D}(T)$. Clearly $E$ is also a c.o.s. in $\mathcal{U}$ and therefore $[\mathcal{U}] = \text{Span}(E)$ is a primitive subspace of $\mathcal{U}$ (see §5.1.1) and a primitive operator $T^1:[\mathcal{U}] \to \mathcal{X}$ of $T$ can be easily defined by

$$T^1u = Tu\ , \qquad u \in [\mathcal{U}]\ .$$

In this section we present two different methods for obtaining a complete matched set $\mathcal{M}$ for $T^1$. Both methods require that we have $\{\bar{e}_n\}$ and $\{\bar{\phi}_n\}$, c.o.s. in the spaces $\mathcal{D}(T)$ and $\mathcal{X}$ respectively. We have seen in sections 5.6 and 5.7 that once we have $\mathcal{M}$ we can define spaces $\mathcal{U}' = \mathcal{U}'(e_n,\alpha_n,\Gamma)$ and $\mathcal{X}' = \mathcal{X}'(\phi_n,\beta_n,\Lambda)$ by manipulating the numbers $\alpha_n$ and/or $\beta_n$, such that $T$ regarded from $\mathcal{U}'$ to $\mathcal{X}'$ will have some desired topological properties.

In the first method $\mathcal{M} = (\{e_n\}_{n\in\Gamma}, \{\phi_n\}_{n\in\Lambda}, \Delta,\Gamma,\Lambda)$ is generated such that $\{\phi_n\}_{n\in\Lambda}$ is a c.o.s. in $\mathcal{X}$, so that we can obtain $\mathcal{X}' \simeq \mathcal{X}$ by setting

$$\beta_n = \beta = \text{constant} \qquad\qquad \text{for all} \quad n \in \Lambda$$

(or also $(X', \|\cdot\|_{X'}) = (X, \|\cdot\|_X)$ if we set $\beta_n = 1$ for all $n \in \Lambda$) .

In the second method $M = (\{\hat{e}_n\}_{n\in\Gamma}, \{\hat{\phi}_n\}_{n\in\Lambda}, \Delta, \Gamma, \Lambda)$ is generated such that $\{\hat{e}_n\}$ is a c.o.s. in $U$ . In this case we can preserve the original space $U$ with an equivalent norm by setting

$$\alpha_n = \alpha = \text{constant} \qquad\qquad \text{for all} \quad n \in \Gamma$$

(or also preserve $U$ with its norm by setting $\alpha_n = 1$ for all $n \in \Gamma$) .

5.8.1 - <u>First method:</u>

Let $T: U \to X$ be linear and densely defined and $\{\bar{e}_n\}_{n=1,2,\dots}$ and $\{\bar{\phi}_n\}_{n=1,2,\dots}$ be c.o.s. in $\mathcal{D}(T)$ and $X$ respectively. Set

$$\Gamma = \Lambda = \{\pm 1, \pm 2, \dots\} \qquad \text{and}$$

$$\Delta = \mathbb{N} = \{1, 2, \dots\} \quad .$$

Clearly $\Delta \subset \Gamma \cap \Lambda$ and the sets $\Gamma'$ and $\Lambda'$ become

$$\Gamma' = \Lambda' = \{-1, -2, \dots\} \quad .$$

It is easy to verify that if we apply the generalized Gram-Schmidt process to the sequence $\{\xi_n\}_{n=1,2,\dots}$ given by

$$\xi_n = T\bar{e}_n \qquad \text{for} \quad n = 1,2,\dots \tag{5.8.1}$$

then we obtain a c.o.s. in $R(T)$ . In the following procedure we apply the generalized Gram-Schmidt process to $\{\xi_n\}_{n=1,2,\ldots}$, generating $\{\phi_n\}_{n\in\Delta}$ in $X$ and at the same time we generate another sequence $\{e_n\}_{n\in\Delta}$ in $U$.

Define $\{\psi_k\}_{k=1,2,\ldots}$ and $\{f_k\}_{k=1,2,\ldots}$ by

$$\psi_k = \begin{cases} 0 & \text{if } \tilde{\phi}_k = 0 \\ \dfrac{\tilde{\phi}_k}{\|\tilde{\phi}_k\|_X} & \text{if } \tilde{\phi}_k \neq 0 \end{cases} \qquad f_k = \begin{cases} 0 & \text{if } \tilde{\phi}_k = 0 \\ \dfrac{\tilde{e}_k}{\|\tilde{e}_k\|_U} & \text{if } \tilde{\phi}_k \neq 0 \end{cases}$$

where

$$\tilde{\phi}_1 = \xi_1 \qquad \tilde{e}_1 = \bar{e}_1$$

$$\tilde{\phi}_k = \xi_k - \sum_{i=1}^{k-1} \langle \xi_k, \psi_i \rangle_X \psi_i \qquad \tilde{e}_k = \bar{e}_k - \sum_{i=1}^{k-1} \langle \xi_k, \psi_i \rangle_U f_k .$$

It is not difficult to verify that $Tf_k = \psi_k$ for $k = 1,2,\ldots$

Now take $\{\phi_n\}_{n\in\Delta}$ the subsequence of $\{\psi_k\}_{k=1,2,\ldots}$ obtained by eliminating all $\psi_k = 0$ and similarly take $\{e_n\}_{n\in\Delta}$ the subsequence obtained from $\{f_k\}_{k=1,2,\ldots}$ by also eliminating all $f_k = 0$. It is clear that $\{e_n\}_{n\in\Delta}$ and $\{\phi_n\}_{n\in\Delta}$ satisfy

$$Te_n = \phi_n \qquad \text{for all } n \in \Delta \tag{5.8.2}$$

$\{\phi_n\}_{n\in\Delta}$ is a c.o.s. in $R(T)$

$\{e_n\}_{n\in\Delta}$ is linearly independent and

$$\overline{\mathrm{Span}\{e_n\}_{n\in\Delta}} = N(T)^{\perp} . \tag{5.8.3}$$

The next step is to find a completion for $\{\phi_n\}_{n\in\Delta}$ in $X$ and a completion for $\{e_n\}_{n\in\Delta}$ in $U$ (see section 5.4). Using $\{\bar{\phi}_n\}_{n=1,2,\ldots}$, since $\{\phi_n\}_{n\in\Delta}$ is orthonormal, we can apply the method of theorem 5.4.2 straight away obtaining a sequence, say $\{\tau_n\}_{n\in \mathbb{N}}$, which is a completion for $\{\phi_n\}_{n\in\Delta}$ in $X$. Setting

$$\phi_{-n} = \tau_n \qquad \text{for } n \in \mathbb{N}$$

we get $\{\phi_n\}_{n\in\Lambda'}$. Clearly we now have that

$$\{\phi_n\}_{n\in\Lambda} \quad \text{is a c.o.s. in } X \tag{5.8.4}$$

and this implies that

$$[X] = \mathrm{Span}\{\phi_n\}_{n\in\Lambda} \tag{5.8.5}$$

is a primitive subspace of $X$. Similarly, using $\{\bar{e}_n\}_{n=1,2,\ldots}$ we can obtain a completion, say $\{\theta_n\}_{n\in\mathbb{N}}$, for the linearly independent set $\{e_n\}_{n=1,2,\ldots}$ (see §5.4.2). Setting

$$e_{-n} = \theta_n \qquad \text{for } n \in \mathbb{N}$$

we get $\{e_n\}_{n\in\Gamma'}$. Equations (5.4.6) and (5.8.2) yield

$$\overline{\mathrm{Span}\{e_n\}_{n\in\Gamma'}} = \overline{N(T)}$$

which implies that

$$Te_n = 0 \qquad \text{for all} \quad n \in \Gamma' = \Gamma\backslash\Delta \tag{5.8.6}$$

and

$$[U] = \text{Span}\{e_n\}_{n\in\Gamma} \tag{5.8.7}$$

is a primitive subspace of $U$ .

From (5.8.2) and (5.8.6) the set

$$M = (\{e_n\}_{n\in\Gamma} , \{\phi_n\}_{n\in\Lambda},\Delta,\Gamma,\Lambda)$$

is a matched set for the primitive operator $T^1:[U] \to X$ of $T$ given by

$$T^1u = Tu , \qquad u \in [U]$$

and by (5.8.5) and (5.8.7) $M$ is complete. Moreover, $M$ satisfies (5.8.4).

5.8.2 - <u>Second Method</u>:

Let $T$, $\{\bar{e}_n\}_{n=1,2,\ldots}$, $\{\bar{\phi}_n\}_{n=1,2,\ldots}$ , $\Delta$, $\Gamma$ and $\Lambda$ be as in the first method. Suppose we have obtained the sets $\{e_n\}_{n\in\Delta}$ and $\{\phi_n\}_{n\in\Lambda'}$ as in the first method. Define $\{\hat{\phi}_n\}_{n\in\Lambda'}$ by

$$\hat{\phi}_n = \phi_n \qquad \text{for all} \quad n \in \Lambda' = \Lambda\backslash\Delta \ . \tag{5.8.8}$$

Clearly we have

$$\overline{\text{Span}\{\hat{\phi}_n\}_{n\in\Lambda'}} = R(T)^\perp \ . \tag{5.8.9}$$

Applying the Gram-Schmidt process to $\{e_n\}_{n\in\Delta}$ we obtain an orthonormal set $\{\hat{e}_n\}_{n\in\Delta}$ which satisfies

$$\overline{\mathrm{Span}\{\hat{e}_n\}_{n\in\Delta}} = N(T)^{\perp} \quad . \tag{5.8.10}$$

Now we obtain $\{\theta_n\}_{n\in \mathbb{N}}$, a completion for $\{\hat{e}_n\}_{n\in\Delta}$, by applying the method of theorem 5.4.2 straight away, using c.o.s. $\{\bar{e}_n\}_{n=1,2,\ldots}$ . Setting

$$\hat{e}_{-n} = \theta_n \qquad\qquad \text{for all} \quad n \in \mathbb{N}$$

we get $\{\hat{e}_n\}_{n\in\Gamma'}$, an orthonormal set in $N(T)$ which satisfies

$$\overline{\mathrm{Span}\{\hat{e}_n\}_{n\in\Gamma'}} = \overline{N(T)} \ . \tag{5.8.11}$$

Also,

$$\{e_n\}_{n\in\Gamma} \quad \text{is a c.o.s. in } U \tag{5.8.12}$$

and

$$T\hat{e}_n = 0 \qquad\qquad \text{for all} \quad n \in \Gamma' = \Gamma\backslash\Delta \ . \tag{5.8.13}$$

Define $\{\hat{\phi}_n\}_{n\in\Delta}$ by

$$\hat{\phi}_n = T\hat{e}_n \qquad\qquad \text{for all} \quad n \in \Delta \ . \tag{5.8.14}$$

Then, by (5.8.10), it is easy to see that

$\{\hat{\phi}_n\}_{n\in\Delta}$ is a linearly independent set and $\overline{\mathrm{Span}\{\hat{\phi}_n\}_{n\in\Delta}} = \overline{R(T)}$ and hence, by (5.8.9),

$$[X] = \mathrm{Span}\{\hat{\phi}_n\}_{n\in\Lambda} \tag{5.8.15}$$

is a primitive subspace of $X$. Similarly, by (5.8.10) and (5.8.11),

$$[U] = \mathrm{Span}\{\hat{e}_n\}_{n\in\Gamma} \tag{5.8.16}$$

is a primitive subspace of $U$. Clearly $T^1:[U] \to X$ given by

$$T^1 u = Tu \ , \qquad u \in [U]$$

is a primitive operator of $T$. By (5.8.13) and (5.8.14)

$$M = (\{\hat{e}_n\}_{n\in\Gamma}, \{\hat{\phi}_n\}_{n\in\Lambda}, \Delta, \Gamma, \Lambda)$$

is a matched set for $T^1$ and, by (5.8.15) and (5.8.16), $M$ is complete. Moreover, $M$ satisfies (5.8.12).

5.8.3 - Remarks:

i) Clearly the approach via the generalized Gram-Schmidt process was necessary since for a given c.o.s. $\{\bar{e}_n\}_{n\in\mathbb{N}}$ the set $\{\xi_n\}_{n\in\mathbb{N}}$ defined in (5.8.1), namely

$$\xi_n = T\bar{e}_n \ , \qquad n \in \mathbb{N}$$

is not linearly independent in general.

ii) Roughly speaking both methods involve the same amount of computation. Whereas in the first method the completion of $\{\phi_n\}_{n\in\Delta}$ follows by applying the method of theorem 5.4.2 straight away, the completion of $\{\hat{e}_n\}_{n\in\Delta}$ in the second method follows by applying the same method straight away.

iii) There are a lot of obvious particularizations. Apart from the cases where either $U$ or $X$ (or both) are finite dimensional, there are also the cases where $R(T)$ has finite dimension or finite co-dimension (i.e., $\dim(X \setminus R(T)) < \infty$) and/or $N(T)^{\perp}$ has finite dimension or finite co-dimension (i.e., $\dim(U \setminus N(T)^{\perp}) < \infty$) .

□

## 5.9 - SOME APPLICATIONS.

We shall use the theory developed in this chapter in applications in control where $\mathcal{U} = H^r(0,T;U)$ for some $r \in \mathbb{R}$ (or $\mathcal{U} = H_0^r(0,T;U)$ for some $r \geq 0$) will be the space of input functions and $X = M^s(0,T;Z) = H^s(0,T;Z) \times Z$ will be the space which contains the pair $(z(\cdot),z_f)$ consisting of the trajectory $z(\cdot) \in H^s(0,T;Z)$ and the final state $z_f \in Z$ (the state space). In section 5.10 we present some examples which show how the spaces $\mathcal{U}$ and $X$ can be adjusted. More specific examples will be given in section 6.7.

We shall again use the abbreviation c.o.s. for complete orthonormal set and the symbol "$\simeq$" (introduced in §5.7.7) to write $X \simeq x$ if $\|\cdot\|_X$ is equivalent to $\|\cdot\|_x$ .

### 5.9.1. The spaces $\mathcal{U} = H^r(0,T;U)$ and $\mathcal{U} = H_0^r(0,T;U)$:

Let $\{f_n\}_{n=1,2,\dots}$ be a c.o.s. in a Hilbert space $U$ and consider the spaces

$$\mathcal{U} = \mathcal{U}'(e_{nm} , \alpha_{nm} , \Gamma) . \tag{5.9.1}$$

where $e_{nm}:[0,T] \to U$ and $\alpha_{nm} > 0$ for all $(n,m) \in \Gamma$ . Let us first set

$$e_{nm}(t) = \sin(\frac{m\pi t}{T}).f_n \qquad \text{for all } (n,m) \in \Gamma = \mathbb{N} \times \mathbb{N} .$$

Since $\{\sqrt{2}\,\sin(m\pi/T)t\}_{m=1,2,\ldots}$ is a c.o.s. in $L^2(0,T) = H^0(0,T)$ we can easily deduce, by the imbedding corollary 5.7.8, that for $r \geq 0$ :

If $\alpha_{nm} = m^{2r}$ for all $(n,m) \in \Gamma$ , then $\mathcal{U} \simeq H_0^r(0,T;U)$ .

If $\alpha_{nm} = m^{-2r}$ for all $(n,m) \in \Gamma$ , then $\mathcal{U} \simeq H^{-r}(0,T;U)'$ .

For example,

if $\alpha_{nm} = 1$ for all $(n,m) \in \Gamma$ => $\mathcal{U} = L^2(0,T;U)$ ,

if $\alpha_{nm} = m^2$ for all $(n,m) \in \Gamma$ => $\mathcal{U} \simeq H_0^1(0,T;U)$ ,

if $\alpha_{nm} = 1/m^2$ for all $(n,m) \in \Gamma$ => $\mathcal{U} \simeq H^{-1}(0,T;U)$ ,

etc. The symbol $\simeq$ must be used (except in the case $s = 0$ where $\mathcal{U}$ coincide with $L^2 = H^0$) because in general $\{e_{nm}\}$ will not be orthonormal in $H^r$ (or $H_0^r$) . For example if $\alpha_{nm} = 1/m^2$ then, $\mathcal{U} \simeq H^{-1}(0,T;U)$ , $\{me_{nm}\}$ is a c.o.s. in $\mathcal{U}$ (since $1/\sqrt{\alpha}_{nm} = m$) , whereas $\{m\pi e_{nm}\}$ are the orthonormal elements with unitary norm in $H^{-1}(0,T;U)$. Of course it is possible to choose the right $\alpha_{nm}$ so that we have $\mathcal{U} = H^r$ for some $r \in \mathbb{R}$ rather than $\mathcal{U} \simeq H^r$ , however we shall not be worried with this here. For us $\mathcal{U} \simeq H^r$ or $\mathcal{U} \simeq H_0^r$ will be enough.

Now consider another example. Let $\Gamma = \mathbb{N} \times \mathbb{Z}^+$ , $\mathbb{N} = \{1,2,\ldots\}$ , $\mathbb{Z}^+ = \{0,1,2,\ldots\}$ and set

$$e_{nm}(t) = \cos(\tfrac{m\pi t}{T}).f_n \qquad \text{for} \quad (n,m) \in \Gamma \quad .$$

(Note that for $m = 0$ we have $e_{nm}(t) = e_{no}(t) = f_n$, independent on $t$.) Since $\{\sqrt{2}\cos(m\pi/T)t\}_{m=1,2,\dots} \cup \{1\}$ is a c.o.s. in $L^2(0,T) = H^0(0,T)$ we can deduce, using the imbedding corollary 5.7.8 again, that for $r \geq 0$ we have

$$\left.\begin{array}{lll} \alpha_{no} = 1 & \text{for all} & n \in \mathbb{N} \\ \alpha_{nm} = m^{2r} & \text{for all} & (n,m) \in \mathbb{N}\times\mathbb{N} \end{array}\right\} \Rightarrow \quad \mathcal{U} \simeq H^r(0,T;U) .$$

Here again it is convenient to consider $\mathcal{U} \simeq H^r$ rather than $\mathcal{U} = H^r$. For example, if we want $\mathcal{U} = H^2(0,T;U)$ we must set

$$\alpha_{no} = 1 \,,\; n \in \mathbb{N} \qquad \text{and} \qquad \alpha_{nm} = \tfrac{1}{2}\,(1+m^2\pi^2 + m^4\pi^4) \qquad \text{for } (n,m) \in \mathbb{N}\times\mathbb{N}$$

whereas if we accept $\mathcal{U} \simeq H^2(0,T;U)$ is enough to set $\alpha_{no} = 1$ and $\alpha_{nm} = m^4$.

Now suppose that $\alpha_{nm}$ also depends on $n$. For example, for some constants $\alpha'_n > 0$, $n \in \mathbb{N}$ and some $r \geq 0$ set

$$\begin{array}{lll} \alpha_{no} = \alpha'_n & \text{for all} & n \in \mathbb{N} \\ \alpha_{nm} = \alpha'_n \,.\, m^{2r} & \text{for all} & (n,m) \in \mathbb{N}\times\mathbb{N} \end{array}$$

then $\mathcal{U} \simeq H^r(0,T;U')$ where $U' = \mathcal{U}'(f_n,\alpha'_n,\mathbb{N})$, i.e.

$$U' = \{\sum_{n=1}^{\infty} u_n f_n \;:\; \|u\|^2_{U'} = (\sum_{n=1}^{\infty} |u_n|^2\,\alpha'_n) < \infty\} \tag{5.9.2}$$

Also note that if $\alpha'_n \leq 1$ for all $n \in \mathbb{N}$, then $U' \subset U$ ; if $\alpha'_n = 1$ for all $n \in \mathbb{N}$, then $U' = U$ and if $\alpha'_n \geq 1$ for all $n \in \mathbb{N}$, then $U' \supset U$ .

Observe that, loosely speaking, the larger $\alpha_{nm}$ is the smoother the spaces $\mathcal{U}$ become. If $\alpha_{nm} = \alpha'_n \alpha''_m$ then the larger $\alpha''_m$ is the smoother (in t) $\mathcal{U}$ becomes. Also, if $U$ is some space of functions with variable $x \in \mathbb{R}^N$ (e.g. $U = L^2(\Omega)$ , $\Omega \subset \mathbb{R}^N$ , a case which arises in control of systems described by partial differential equations), then the larger $\alpha'_n$ is the smoother (in x) $\mathcal{U}$ becomes. These properties will be useful in control applications where a smooth control u is desirable.

Now let $\Gamma = \mathbb{N} \times \mathbb{Z}^+$ again and set

$$e_{nm}(t) = t^m . f_n \qquad \text{for} \quad (n,m) \in \Gamma$$

and set

$$\alpha''_m = \| t^m \|^2_{H^r(0,T)} \qquad \text{for} \quad m = 0,1,2,\ldots .$$

By the imbedding corollary 5.7.8 we have that, for $r \geq 0$

If $\alpha_{nm} = \alpha''_m$ then $\mathcal{U} \simeq H^r(0,T;U)$

If $\alpha_{nm} = \alpha'_n . \alpha''_m$ then $\mathcal{U} \simeq H^r(0,T;U')$

where $U'$ is given by (5.9.2). The difference is that now the functions

$\{f_n, t\, f_n, t^2 f_n, \ldots\}_{n=1,2,\ldots}$ is an orthogonal set in $\mathcal{U}$ .
A typical element (control) $u \in \mathcal{U}$ here has the polynomial form

$$u(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} u_{nm} t^m . f_n = \sum_{n=1}^{\infty} u_{no} f_n + u_{n1} t f_n + u_{n2} t^2 f_n + \ldots .$$

Similar results can be obtained with

$$e_{nm} = e'_m f_n \qquad (n,m) \in \Gamma \tag{5.9.3}$$

where $\{e'_m\}$ is some set of linearly independent functions in $L^2(0,T)$ .

Finally, suppose that $\{f_n\}_{n=1,2,\ldots}$ is now a linearly independent set (not necessarily orthogonal) such that Span $\{f_n\}_{n=1,2,\ldots} = U$ and consider $U'$ given by (5.9.2). By the imbedding corollary 5.7.8 we have that if $\alpha'_n = \| f_n \|_U^2$ for $n = 1,2,\ldots$ then $U' \simeq U$ ; if $\alpha'_n > \| f_n \|_U^2$ (respectively $\alpha'_n < \| f_n \|_U^2$) for $n = 1,2,\ldots$ then $U' \subset U$ (respectively $U' \supset U$) . Here too we can obtain similar result as above. For example, let $e_{nm}$ be given by (5.9.3) then, for $r \geq 0$ :

$$\alpha_{nm} = \| e'_m \|^2_{H^r(0,T)} \| f_n \|_U^2 \qquad \Rightarrow \qquad \mathcal{U} \simeq H^r(0,T;U)$$

and

$$\alpha_{nm} = \| e'_m \|^2_{H^r(0,T)} . \alpha'_n \qquad \Rightarrow \qquad \mathcal{U} \simeq H^r(0,T;U') .$$

The above examples show the flexibility we have to represent spaces of the type $H^r$ and $H_0^r$ in the form $\mathcal{U}'(e_{nm}, \alpha_{nm}, \Gamma)$ introduced in

§5.6.1. In the next paragraph we study the representation of spaces of the type $M^s$ (introduced in §2.2.6) in the form $X'(\Phi_k,\beta_k,\Lambda)$ .

5.9.2 - The spaces $X = M^s(0,T;Z)$:

Consider the spaces $M^s(0,T;Z)$ introduced in §2.2.6, namely

$$M^s(0,T;Z) = H^s(0,T;Z) \times Z \ , \qquad s \in \mathbb{R}$$

Z Hilbert space (the state space) and

$$\| z \|^2_{M^s} = \| z(\cdot) \|^2_{H^s} + \| z_f \|^2_Z \qquad \text{for} \quad z = (z(\cdot),z_f) \in M^s(0,T;Z) \ .$$

We shall see here three examples of how to express these spaces in the form $X'(\Phi_j,\beta_j,\Lambda)$ . Here we use $\Phi_j,\Phi_{nm}$ , etc., to denote elements of $M^s$ ; $\Phi_j(\cdot)$ , $\Phi_{nm}(\cdot)$ , etc., to denote elements of $H^s$ and $\phi_j,\phi_{nm}$, etc., to denote elements of $Z$ . For example,

$$\Phi_j = (\Phi_j(\cdot),\phi_j) \in M^s(0,T;Z) \qquad \text{if} \quad \Phi_j(\cdot) \in H^s(0,T;Z) \quad \text{and} \quad \phi_j \in Z \ .$$

i) Let $\Lambda_1$ and $\Lambda_2$ be two countable sets, $\{\Phi_j(\cdot)\}_{j\in\Lambda_1}$ and $\{\phi_n\}_{n\in\Lambda_2}$ any two linearly independent sets such that

$$\{\Phi_j(\cdot)\}_{j\in\Lambda_1} \quad \text{is complete in} \quad H^s(0,T;Z) \qquad \text{and}$$

$$\{\phi_n\}_{n\in\Lambda_2} \qquad \text{is complete in} \quad Z$$

(see section 5.4 for definition of completeness in the above sense).
For example, $\{\phi_n\}_{n\in\Lambda_2}$ any c.o.s. in Z and $\{\Phi_j(\cdot)\}_{j\in\Lambda_1}$ such that

$$\overline{\mathrm{Span}\{\Phi_j(\cdot)\}_{j\in\Lambda_1}} = H^s(0,T;Z)$$

where the bar denotes completion with respect to the norm in $H^s(0,T;Z)$.
$e'_m(t) = \cos(m\pi/T)t$ or $e'_m(t) = t^{m-1}$ , etc. See §5.9.1.) Set

$$E_1 = X'((\Phi_j(\cdot),0) \ , \ \beta'_j \ , \Lambda_1)$$

$$E_2 = X'((0,\phi_n) \ , \ \beta''_n \ , \Lambda_2)$$

and

$$X = E_1 \oplus E_2 \ .$$

By the imbedding corollary 5.7.8 we have that: for $s \in \mathbb{R}$ if

$$\beta'_j = \| \Phi_j(\cdot) \|^2_{H^s} \ , \quad j \in \Lambda_1 \qquad \text{and} \qquad \beta''_n = \| \phi_n \|^2_Z \ , \quad n \in \Lambda_2$$

then

$$E_1 \simeq H^s(0,T;Z) \times \{0\} \ , \qquad E_2 = \{0\} \times Z \tag{5.9.4}$$

$$X \simeq M^s(0,T;Z) \ .$$

Clearly, by (5.9.4),

$$H^s(0,T;Z) \simeq s_1(E_1) = s_1(X)$$

and

$$Z \simeq s_2(E_2) = s_2(X)$$

where $s_1$ and $s_2$ are the submersions defined in $X$ by

$$s_1(z) = z(\cdot) \qquad \text{for } z = (z(\cdot), z_f) \in X ,$$

$$s_2(z) = z_f \qquad \text{for } z = (z(\cdot), z_f) \in X .$$

Summarizing, any $z = (z(\cdot), z_f) \in M^s(0,T;Z)$ can be written as

$$z = \sum_{j\in\Lambda_1} a_j(\Phi_j(\cdot),0) + \sum_{n\in\Lambda_2} b_n(0,\phi_n) .$$

ii) The above representation of the spaces $M^s$, though simple, will not be the most suitable for applications in chapter 6. Here we present a different way to express $M^s(0,T;Z)$. Let $\Delta$ be a countable set and suppose we have a linearly independent set R

$$R = \{\Phi_j\}_{j\in\Delta} \tag{5.9.5}$$

of elements $\Phi_j = (\Phi_j(\cdot), \phi_j) \in M^s(0,T;Z)$ for some $s \in \mathbb{R}$. We first assume that

$$\{\Phi_j(\cdot)\}_{j\in\Delta} \qquad \text{is linearly independent} \tag{5.9.6}$$

and

$$\left\{ \frac{\| \Phi_j(\cdot) \|_{H^s}}{\| \phi_j \|_Z} : \phi_j \neq 0 \right\}_{j\in\Delta} \qquad \text{is bounded.} \tag{5.9.7}$$

(Observe that if (5.9.6) holds but (5.9.7) does not hold then we can easily redefine $R$ as

$$\tilde{R} = \{\tilde{\Phi}_j\}_{j\in\Delta}$$

where $\tilde{\Phi}_j = (\Phi_j(\cdot), c_j\phi_j)$ for $j \in \Delta$ and

$$c_j = \begin{cases} \| \Phi_j(\cdot) \|_{H^s} / \| \phi_j \|_Z & \text{if} \quad \phi_j \neq 0 \\ 0 & \text{if} \quad \phi_j = 0 . \end{cases}$$

With this definition

$$\mathrm{Span}(\tilde{R}) = \mathrm{Span}(R)$$

and $\tilde{R}$ satisfies both (5.9.6) and (5.9.7).)

Let $\Lambda_0, \Lambda_1, R_0$ and $R_1$ be the sets

$$R_0 = \{\Phi_j \in R : \phi_j = 0\} \quad , \qquad \Lambda_0 = \{j \in \Delta : \phi_j = 0\} ,$$

$$R_1 = \{\Phi_j \in R : \phi_j \neq 0\} \quad , \qquad \Lambda_1 = \{j \in \Delta : \phi_j \neq 0\} .$$

Clearly $R = R_0 \cup R_1$ and $\Delta = \Lambda_0 \cup \Lambda_1$ . Now set

$$\Phi_{0j}(\cdot) = \Phi_j(\cdot) \qquad\qquad \text{for} \quad j \in \Lambda_0$$

$$\Phi_{1j}(\cdot) = \Phi_j(\cdot) \quad \text{and} \quad \phi_{1j} = \phi_j \qquad \text{for} \quad j \in \Lambda_1$$

and take $\{\Phi_{2j}(\cdot)\}_{j\in\Lambda_2}$ and $\{\phi_{4j}\}_{j\in\Lambda_4}$ such that (see section 5.4)

$$\{\Phi_{2j}(\cdot)\}_{j\in\Lambda_2} \quad \text{is a completion of} \quad \{\Phi_j(\cdot)\}_{j\in\Delta} \quad \text{in} \quad H^s(0,T;Z)$$

and

$$\{\phi_{4j}\}_{j\in\Lambda_4} \quad \text{is a completion of} \quad \{\phi_j\}_{j\in\Delta} \quad \text{in} \quad Z .$$

Clearly either $\Lambda_2$ or $\Lambda_4$ may be empty sets. Actually $\Lambda_2$ will be empty if $\overline{\text{Span}\{\Phi_j(\cdot)\}}_{j\in\Delta} = H^s(0,T;Z)$ and similarly, $\Lambda_4$ will be empty if $\overline{\text{Span}\{\phi_j\}}_{j\in\Delta} = \overline{\text{Span}\{\phi_{1j}\}}_{j\in\Lambda_1} = Z$ (see section 5.4).

Now define $\Lambda_3 = \Lambda_1$ ,

$$\begin{aligned}
\Phi_{0j} &= (\Phi_{0j}(\cdot) \ , \ 0 \ ) && j \in \Lambda_0 \\
\Phi_{1j} &= (\Phi_{1j}(\cdot) \ , \ \phi_{1j}) && j \in \Lambda_1 \\
\Phi_{2j} &= (\Phi_{2j}(\cdot) \ , \ 0 \ ) && j \in \Lambda_2 \\
\Phi_{3j} &= (\Phi_{1j}(\cdot) \ , \ 0 \ ) && j \in \Lambda_3 \\
\Phi_{4j} &= ( \ 0 \ , \ \phi_{4j}) && j \in \Lambda_4
\end{aligned} \tag{5.9.8}$$

and

$$\beta_{ij} = \|\Phi_{ij}\|^2_{M^s} \ , \qquad j \in \Lambda_i \ , \quad i \in \{0,1,2,3,4\} \ . \tag{5.9.9}$$

Also set

$$E_i = X'(\Phi_{ij} \ , \ \beta_{ij} \ , \ \Lambda_i) \qquad i \in \{0,1,2,3,4\}$$

and

$$X = \sum_{i=0}^{4} E_i = E_0 \oplus E_1 \oplus E_2 \oplus E_3 \oplus E_4$$

with norm defined by

$$\| z \|_X = ( \sum_{i=0}^{4} \| z^i \|_{E_i}^2 )^{\frac{1}{2}} \tag{5.9.10}$$

for $z = z^0 + z^1 + \ldots + z^4$ , $z^i \in E_i$ , $i = 0,1,2,3,4$ . It is easy to verify (by the imbedding corollary 5.7.8)

$$X \simeq M^s(0,T;Z) \; .$$

Moreover,

$$E_i \perp E_j \qquad \text{for} \quad i \neq j \quad i,j \in \{0,1,2,3,4\} \tag{5.9.11}$$

$$E_0 = \overline{\mathrm{Span}(R_0)} \; , \qquad E_1 = \overline{\mathrm{Span}(R_1)} \qquad \text{and} \qquad E_0 \oplus E_1 = \overline{\mathrm{Span}(R)} . \tag{5.9.12}$$

Clearly

$$E_0 \oplus E_2 \oplus E_3 = H^s(0,T;Z) \times \{0\}$$

and therefore $\| \cdot \|_{H^s} \simeq \| (\cdot,0) \|_X$ and

$$H^s(0,T;Z) \simeq s_1(E_0 \oplus E_2 \oplus E_3) \; .$$

Now let $Z_1$ , $Z_2$ and $\hat{Z}$ be the spaces

$$Z_1 = s_2(E_1) \quad , \qquad Z_2 = s_2(E_4) \qquad \text{and} \qquad \hat{Z} = s_2(E_1 \oplus E_4) = s_2(X) \; . \tag{5.9.13}$$

Clearly

$$\hat{Z} = Z_1 \oplus Z_2 \; .$$

Also note that

$$\hat{Z} \simeq Z$$

since the sets

$$\{\beta_{ij}/\|\phi_{ij}\|_Z^2\}_{j\in\Lambda_i} \quad \text{and} \quad \{\|\phi_{ij}\|_Z^2/\beta_{ij}\}_{j\in\Lambda_i} \qquad i = 1,4$$

are all bounded, which is guaranteed by (5.9.7). As a matter of fact $\hat{Z}$ is the space $Z$ with an equivalent topology such that

$$\{\phi_{ij} \,/\, \sqrt{\beta_{ij}} : j \in \Lambda_i \;,\; i = 1,4\} \qquad \text{is a c.o.s. in } \hat{Z} \,.$$

Moreover,

$$\|z_f\|_{\hat{Z}} = \|(0,z_f)\|_X \quad .$$

Note that if $\Lambda_4 = \emptyset$ (empty) then $Z_2$ is empty. This will be the case if and only if $\text{Span}\{\phi_j\}_{j\in\Delta} = \text{Span}\{\phi_{1j}\}_{j\in\Lambda_1} = Z$ . If $Z_2 \neq \emptyset$ then $Z_2$ is the closed subspace of $Z$ given by

$$Z_2 = X'(\phi_{2j} \;,\; \beta_{4j} \;,\; \Lambda_4) \;.$$

If

$$\{\phi_j\}_{j\in\Lambda_1} \quad \text{is linearly independent}$$

then $Z_1$ can also be expressed as

$$Z_1 = X'(\phi_j \;,\; \beta_{ij} \;,\; \Lambda_1) \;.$$

Summarizing, if $z = (z(\cdot), z_f) \in M^s(0,T;Z)$ for some $s \in \mathbb{R}$ then $z$ can be expressed as

$$z = \sum_{i=0}^{4} \sum_{j\in\Lambda_i} a_{ij} \Phi_{ij}$$

where $a_{ij} \in \mathbb{F}$ and $\Phi_{ij}$ is given by (5.9.8). If $\beta_{ij}$ is given by (5.9.9) then the norm $\|\cdot\|_X$, given by

$$\| z \|_X^2 = \sum_{i=0}^{4} \sum_{j\in\Lambda_i} |a_{ij}|^2 \beta_{ij}$$

is equivalent to $\|\cdot\|_{M^s}$. Furthermore,

$$\left\{ \frac{\Phi_{ij}}{\sqrt{\beta_{ij}}} : j \in \Lambda_i \; ; \quad i = 0,1,2,3,4 \right\} \qquad \text{is a c.o.s. in } X ,$$

the orthogonal projection $P_i : X \to X$ onto $E_i$ is given by

$$P_i z = \sum_{j\in\Lambda_i} a_{ij} \Phi_{ij} , \qquad i = 0,1,2,3,4 \tag{5.9.14}$$

and the orthogonal projection $P_R : X \to X$ onto $\overline{\text{Span}(R)} = E_0 \oplus E_1$ is given by

$$P_R z = \sum_{i=0,1} \sum_{j\in\Lambda_i} a_{ij} \Phi_{ij} . \tag{5.9.15}$$

Finally, if (5.9.6) does not hold then we have to find $\{\Phi'_{1j}(\cdot)\}_{j\in\Lambda_3}$ linearly independent such that

$$\mathrm{Span}\{\Phi'_{1j}(\cdot)\}_{j\in\Lambda_3} = \mathrm{Span}\{\Phi_{1j}(\cdot)\}_{j\in\Lambda_1} .$$

This can be done by using the technique developed in section 5.3, the generalized Gram-Schmidt process. If we replace $\Phi_{3j}$ and $\beta_{3j}$ by

$$\Phi'_{3j} = (\Phi'_{1j}(\cdot), 0) \qquad \beta'_{3j} = \| \Phi'_{3j} \|^2_{M^S} = \| \Phi'_{1j}(\cdot) \|^2_{H^S} \qquad j \in \Lambda_3$$

then $E_3$ becomes

$$E_3 = X'(\Phi'_{3j} , \beta'_{3j} , \Lambda_3)$$

and the above results remain valid.

iii) Here we consider another way to express $M^S(0,T;Z)$ . Let $R$ be as defined in (5.9.5) but instead of conditions (5.9.6) and (5.9.7), here we assume

$$\{\phi_j\}_{j\in\Delta} \quad \text{is linearly independent} \tag{5.9.16}$$

and

$$\{ \frac{\| \phi_j \|_Z}{\| \Phi_j(\cdot) \|_{H^S}} : \Phi_j(\cdot) \neq 0\}_{j\in\Delta} \quad \text{is bounded.} \tag{5.9.17}$$

(Observe that if (5.9.16) holds but (5.9.17) does not hold then we can easily redefine $R$ as

$$\tilde{R} = \{\tilde{\Phi}_j\}_{j\in\Delta}$$

where $\tilde{\Phi}_j = (c_j\Phi_j(\cdot),\phi_j)$ for $j \in \Delta$ and

$$c_j = \begin{cases} \| \phi_j \|_Z \,/\, \| \Phi_j(\cdot) \|_{H^s} & \text{if } \Phi_j(\cdot) \neq 0 \\ 0 & \text{if } \Phi_j(\cdot) = 0 . \end{cases}$$

Clearly

$$\mathrm{Span}(\tilde{R}) = \mathrm{Span}(R)$$

and $\tilde{R}$ satisfies both (5.9.16) and (5.9.17).)

Let $\Lambda_0$ , $\Lambda_1$ , $R_0$ and $R_1$ be defined by

$$R_0 = \{\Phi_j \in R : \Phi_j(\cdot) = 0\} \qquad \Lambda_0 = \{j \in \Delta : \Phi_j(\cdot) = 0\} ,$$

$$R_1 = \{\Phi_j \in R : \Phi_j(\cdot) \neq 0\} \qquad \Lambda_1 = \{j \in \Delta : \Phi_j(\cdot) \neq 0\} .$$

Clearly $R = R_0 \cup R_1$ and $\Delta = \Lambda_0 \cup \Lambda_1$ . Now set

$$\phi_{0j} = \phi_j \qquad \text{for } j \in \Lambda_0$$

$$\Phi_{1j}(\cdot) = \Phi_j(\cdot) \text{ and } \phi_{1j} = \phi_j \qquad \text{for } j \in \Lambda_1$$

and take $\{\Phi_{2j}(\cdot)\}_{j\in\Lambda_2}$ and $\{\phi_{4j}\}_{j\in\Lambda_4}$ such that (see section 5.4)

$\{\Phi_{2j}\}_{j\in\Lambda_2}$ is a completion of $\{\Phi_j(\cdot)\}_{j\in\Delta}$ in $H^s(0,T;Z)$

and

$\{\phi_{4j}\}_{j\in\Lambda_4}$ is a completion of $\{\phi_j\}_{j\in\Delta}$ in $Z$ .

Clearly either $\Lambda_2$ or $\Lambda_4$ (or both) may be empty (see section 5.4). Now define $\Lambda_3 = \Lambda_1$ and

$$\begin{aligned}
\Phi_{0j} &= (0 \ , \ \phi_{0j}) && j \in \Lambda_0 \\
\Phi_{1j} &= (\Phi_{1j}(\cdot), \ \phi_{1j}) && j \in \Lambda_1 \\
\Phi_{2j} &= (\Phi_{2j}(\cdot), \ 0 \ ) && j \in \Lambda_2 \qquad (5.9.18)\\
\Phi_{3j} &= (0 \ , \ \phi_{1j}) && j \in \Lambda_3 \\
\Phi_{4j} &= (0 \ , \ \phi_{4j}) && j \in \Lambda_4
\end{aligned}$$

and $\beta_{ij}$ as in (5.9.9), i.e.,

$$\beta_{ij} = \| \Phi_{ij} \|^2_{M^s} \ , \qquad j \in \Lambda_i \ , \quad i \in \{0,1,2,3,4\} \ . \qquad (5.9.19)$$

We also define $E_i$ and $X$ similarly to example (ii), i.e.,

$$E_i = X'(\Phi_{ij} \ , \ \beta_{ij} \ , \ \Lambda_i) \qquad i \in \{0,1,2,3,4\}$$

$$X = \sum_{i=0}^{4} E_i = E_0 \oplus E_1 \oplus E_2 \oplus E_3 \oplus E_4$$

with norm $\| \cdot \|_X$ given by (5.9.10). Here too we have that

$$X \simeq M^s(0,T;Z)$$

and expressions (5.9.11) and (5.9.12) hold (i.e., $E_i \perp E_j$ , $i \neq j$ and $i,j = 0,1,2,3,4$; $E_0 = \overline{\mathrm{Span}(R_0)}$ , $E_1 = \overline{\mathrm{Span}(R_1)}$ and $E_0 \oplus E_1 = \overline{\mathrm{Span}(\dot{R})}$).

Now let $F_1$ , $F_2$ and $F$ be the spaces

$$F_1 = s_1(E_1) \quad , \quad F_2 = s_1(E_2) \qquad \text{and} \qquad F = s_1(E_1 \oplus E_2) = s_1(X)$$

then

$$F = F_1 \oplus F_2$$

and

$$F \simeq H^s(0,T;Z) \ .$$

This follows since the sets

$$\{\beta_{ij} / \| \Phi_{ij}(\cdot) \|^2_{H^s}\}_{j \in \Lambda_i} \qquad \text{and} \qquad \{\| \Phi_{ij}(\cdot) \|^2_{H^s} / \beta_{ij}\}_{j \in \Lambda_i} \qquad i = 1,2$$

are all bounded, which is guaranteed by (5.9.17). As a matter of fact $F$ is the space $H^s(0,T;Z)$ with an equivalent topology such that

$$\{\Phi_{ij}(\cdot) / \sqrt{\beta_{ij}} : j \in \Lambda_i \ , \ i = 1,2\} \quad \text{is a c.o.s. in } F \ .$$

Moreover

$$\| z(\cdot) \|_F = \| (z(\cdot),0) \|_X \ .$$

Observe now that

$$E_0 \oplus E_3 \oplus E_4 \simeq \{0\} \times Z$$

and therefore we have $\|\cdot\|_Z \simeq \|(0,\cdot)\|_X$ and

$$Z \simeq s_2(E_0 \oplus E_3 \oplus E_4) .$$

Summarizing, if $z = (z(\cdot), z_f) \in M^s(0,T;Z)$ for some $s \in \mathbb{R}$ then $z$ can be expressed as

$$z = \sum_{i=0}^{4} \sum_{j \in \Lambda_i} a_{ij} \Phi_{ij}$$

where $a_{ij} \in \mathbb{F}$ and $\Phi_{ij}$ is given by (5.9.18). If $\beta_{ij}$ is given (5.9.19) then the norm $\|\cdot\|_X$ , given by

$$\| z \|_X = \sum_{i=0}^{4} \sum_{j \in \Lambda_i} |a_{ij}|^2 \beta_{ij}$$

is equivalent to $\|\cdot\|_{M^s}$ . Moreover,

$$\left\{ \frac{\Phi_{ij}}{\sqrt{\beta_{ij}}} : j \in \Lambda_i \ ; \ i = 0,1,2,3,4 \right\} \text{ is a c.o.s. in } X .$$

The orthogonal projections $P_i : X \to X$ onto $E_i$ , $i = 0,1,2,3,4$ are given by

$$P_i z = \sum_{j \in \Lambda_i} a_{ij} \Phi_{ij} \tag{5.9.20}$$

and the orthogonal projection $P_R : X \to X$ onto $\overline{\text{Span}(R)} = E_0 \oplus E_1$ is given by

$$P_R z = \sum_{i=0,1} \sum_{j \in \Lambda_i} a_{ij} \, \Phi_{ij} \quad . \tag{5.9.21}$$

Finally, if (5.9.16) does not hold then we have to find $\{\phi'_{1j}\}_{j \in \Lambda_3}$ linearly independent such that

$$\text{Span}\{\phi'_{1j}\}_{j \in \Lambda_3} = \text{Span}\{\phi_{1j}\}_{j \in \Lambda_1} \quad .$$

This can be done by applying the generalized Gram-Schmidt process (see section 5.3) to $\{\phi_{1j}\}_{j \in \Lambda_1}$ . Replacing $\Phi_{3j}$ and $\beta_{3j}$ by

$$\Phi'_{3j} = (0, \phi'_{1j}) \qquad \beta'_{3j} = \| \Phi_{3j} \|^2_{M^s} = \| \phi'_{1j} \|^2_Z \qquad j \in \Lambda_3$$

$E_3$ becomes

$$E_3 = X'(\Phi'_{3j} \, , \, \beta'_{3j} \, , \, \Lambda_3)$$

and the above results still hold.

Suppose now that $T$ is some linear operator from $H^r(0,T;U)$ $r \in \mathbb{R}$ (or $H^r_0(0,T;U), r \geq 0$) to $M^s(0,T;Z)$ , $s \in \mathbb{R}$ and

$$R = \{Te_j\}_{j \in \Delta}$$

is linearly independent for some sequence $\{e_j\}_{j \in \Delta}$ in $H^r(0,T;U)$ (or $H^r_0(0,T;U)$) also linearly independent. The results of this paragraph together with theorems 5.7.1 and 5.7.3 and corollary 5.7.5 allow us to deduce the following three theorems:

5.9.3 - Theorem:

Let $T$ be a linear operator from $H^r(0,T;U)$ to $M^s(0,T;Z)$ for some $r \in \mathbb{R}$, $s \in \mathbb{R}$, $\mathcal{U}$ be the space

$$\mathcal{U} = \mathcal{U}'(e_j, \alpha_j, \Delta)$$

for some linearly independent sequence $\{e_j\}_{j\in\Delta}$ in $H^r(0,T;U)$ (see §5.9.1). Define

$$R = \{Te_j\}_{j\in\Delta}$$

and suppose that $R$ is linearly independent. Then we have

$$\alpha_j \geq \|Te_j\|^2_{M^s} \quad \text{for all } j \in \Delta \quad \Rightarrow \quad T \in \mathcal{L}(\mathcal{U}, M^s),$$

$$\alpha_j = \|Te_j\|^2_{M^s} \quad \text{for all } j \in \Delta \quad \Rightarrow \begin{cases} T \in \mathcal{L}(\mathcal{U}, M^s) \text{ and} \\ R(T) \text{ is closed in } M^s, \end{cases}$$

$$\alpha_j \leq \|Te_j\|^2_{M^s} \quad \text{for all } j \in \Delta \quad \Rightarrow \quad R(T) \text{ is closed in } M^s.$$

5.9.4 - Theorem:

Let $T$ be again a linear operator from $H^r(0,T;U)$ to $M^s(0,T;Z)$ for some $r \in \mathbb{R}$, $s \in \mathbb{R}$. Define

$$R = \{Te_j\}_{j\in\Delta}$$

for some linearly independent sequence $\{e_j\}_{j\in\Delta}$ in $H^r(0,T;U)$ and suppose that R is linearly independent. Let $X$ be as constructed in either example (ii) or (iii) in §5.9.2 and set

$$\beta_j = \begin{cases} \beta_{0j} & \text{if} \quad j \in \Lambda_0 \\ \beta_{1j} & \text{if} \quad j \in \Lambda_1 . \end{cases} \tag{5.9.22}$$

Then,

$$\beta_j \le \|e_j\|^2_{H^r} \quad \text{for all} \quad j \in \Delta \quad \Rightarrow \quad T \in \mathcal{L}(H^r, X) ,$$

$$\beta_j = \|e_j\|^2_{H^r} \quad \text{for all} \quad j \in \Delta \quad \Rightarrow \quad \begin{cases} T \in \mathcal{L}(H^r, X) \quad \text{and} \\ R(T) \quad \text{is closed in} \quad X , \end{cases}$$

$$\beta_j \ge \|e_j\|^2_{H^r} \quad \text{for all} \quad j \in \Delta \quad \Rightarrow \quad R(T) \quad \text{is closed in} \quad X .$$

5.9.5 - Theorem:

Let now $T$ be a linear operator from $H^r_0(0,T;U)$ to $M^s(0,T;Z)$ for some $r \ge 0$ and $s \in \mathbb{R}$. Define

$$R = \{Te_j\}_{j\in\Delta}$$

for some linearly independent sequence $\{e_j\}_{j\in\Delta}$ in $H^r_0(0,T;U)$ and supposed that R is linearly independent. If $X$ is as constructed in either example (ii) or (iii) of §5.9.2 and $\beta_j, j\in\Delta$ as in (5.9.22), then

$$\beta_j \leq \| e_j \|^2_{H_0^r} \qquad \text{for all} \quad j \in \Delta \qquad => \qquad T \in \mathcal{L}(H_0^r, X) \; ,$$

$$\beta_j = \| e_j \|^2_{H_0^r} \qquad \text{for all} \quad j \in \Delta \qquad => \qquad \begin{cases} T \in \mathcal{L}(H_0^r, X) & \text{and} \\ R(T) & \text{is closed in} \quad X \; , \end{cases}$$

$$\beta_j \geq \| e_{ij} \|^2_{H_0^r} \qquad \text{for all} \quad j \in \Delta \qquad => \qquad R(T) \quad \text{is closed in} \quad X \; .$$

## 5.10 - APPLICATION IN CONTROL

Consider the nonlinear system (3.2.7), namely

$$z = Sz_0 + LBu(\cdot) + LNz(\cdot)$$

and set

$$S = LB(U_{ad})$$

where we consider two cases for $U_{ad}$ :

$$U_{ad} = U \qquad \text{or} \quad U_{ad} = \text{closed subspace of} \quad U \tag{5.10.1}$$

or

$$U_{ad} \quad \text{is a bounded, closed and convex subset of} \quad U \; . \tag{5.10.2}$$

In chapter 6 we shall develop some applications of the projections (which we studied in chapter 4) to nonlinear controllability and we want to have a continuous projection $P$ onto $S$ . Clearly, by theorem 4.1.3, this is only possible if

$$S \text{ is closed.} \tag{5.10.3}$$

If $\mathcal{U}_{ad}$ is as in (5.10.1) then $S = \mathcal{R}(LB)$ and (5.10.3) is equivalent to

$$LB \text{ has closed range.} \tag{5.10.4}$$

If $\mathcal{U}_{ad}$ is as in (5.10.2) then, by theorem 5.2.4, (5.10.3) also holds if

$$LB \text{ is bounded and has closed range.} \tag{5.10.5}$$

Since LB is a linear operator from $H^r(0,T;U)$ , $r \in \mathbb{R}$ (or $H_0^r(0,T;U)$, $r \geq 0$) to $\mathcal{M}^s(0,T;Z)$ , $s \in R$ we can apply the results of the previous section to determine spaces of input functions $\mathcal{U}$ and $X$ of the type $\mathcal{M}(0,T;Z)$ such that either condition (5.10.4) or (5.10.5) will hold. We shall see some specific examples in section 6.7.

In general, large space of input functions (such as $\mathcal{U} = H^{-1}(0,T;U)$) are not desirable since it may contain distributions. Even the spaces $\mathcal{U} = L^2(0,T;U)$ may sometimes be unsuitable for applications since it contains discontinuous functions. The results of the present chapter allow us to select spaces $\mathcal{U}$ and $X$ such that either condition (5.10.4) or (5.10.5) (depending on the problem) will hold and $\mathcal{U} \simeq$ to some smooth space of functions from $[0,T]$ to U (such as $H^1(0,T;U)$ or $H_0^1(0,T;U)$).

Loosely speaking, the smoother we want $\mathcal{U}$ to be, the smoother $X$ will have to be, that is, we shall have to restrict to smoother trajectories. This will be illustrated in examples in section 6.7.

In the case of infinite dimensional systems the output space $U$ may be infinite dimensional (e.g. $U = L^2(\Omega)$ , $\Omega \subset \mathbb{R}^n$) and it may be interesting if we consider space of input functions $\mathcal{U} \simeq L^2(0,T;U')$ or $\mathcal{U} \simeq H^1(0,T;U')$ where $U'$ is a smoother space than $U$ (e.g., $U' = H^1(\Omega)$). Again, the result of this chapter will allow us to do this too. Here however we may have to admit $X \simeq M^s(0,T;Z')$ with $Z'$ smoother than $Z$ . That is, a new state space $Z'$ smoother than $Z$ .

5.10.1 - $\underline{LB:\mathcal{U} \to X}$ :

The theorems of the previous section will be useful because in most cases, as we shall see in section 6.7, simple sequences of functions $\{e_j\}_{j\in\Delta}$ (such as sines, cosines or monomials) will give

$$R = \{LBe_j\}_{j\in\Delta}$$

linearly independent. With $R$ we can construct the space

$$X = \sum_{i=0}^{4} E_i = E_0 \oplus E_1 \oplus E_2 \oplus E_3 \oplus E_4 \tag{5.10.6}$$

where

$$E_i = X'(\Phi_{ij}, \beta_{ij}, \Lambda_i) \qquad i = 0,1,2,3,4$$

and $\Phi_{ij}$ is given by either (5.9.8) (see example (ii) of §5.9.2) or (5.9.18) (see example (iii) of §5.9.2).

This representation of $X$ has several advantages, for example,

$$\overline{\mathcal{R}(LB)} = E_0 \oplus E_1 = \overline{\mathrm{Span}(R)}$$
$$\mathcal{R}(LB)^{\perp} = E_2 \oplus E_3 \oplus E_4 \ . \tag{5.10.7}$$

and the orthogonal projection onto $\overline{\mathcal{R}(LB)} = \overline{\mathrm{Span}(R)}$ is given by $P_R$ (see examples (ii) and (iii) of §5.9.2).

Let

$$\mathcal{U} = \mathcal{U}'(e_j, \alpha_j, \Delta) \tag{5.10.8}$$

then

$$\mathcal{U} = \mathcal{N}(LB)^{\perp} \qquad \text{and} \qquad \mathcal{N}(LB) = \{0\} \ .$$

The results of this chapter allow us to choose $\alpha_j$ and $\beta_{ij}$ such that either $LB \in \mathcal{L}(\mathcal{U}, X)$ or $\mathcal{R}(LB)$ is closed (in $X$), or both. We shall return to this point in §§5.10.3 and 5.10.4. For the moment suppose that either condition (5.10.4) or (5.10.5) hold.

Consider the projection $\Pi$ defined by (4.3.12) (see example 4.3.6). Actually $\Pi$ is a family of projections since $\Pi = \Pi_1 + \Pi_2$ , $\Pi_1$ given by (4.3.9) and $\Pi_2 = (I - \Pi_1)P$ for any continuous projection $P$ onto $\mathcal{R}(LB)$ . Set $P = P_R$ = orthogonal projection onto $\mathcal{R}(LB)$ given by (5.9.15) (see example (ii) of §5.9.2). Also consider the set $S_1$ given by (4.3.10), namely

$$S_1 = \{LBu \in X \ : \ u \in \mathcal{N}(L(T)B)^{\perp}\} \subset \mathcal{R}(LB) \ .$$

It is easy to verify that,

$$E_1 = S_1$$

$$\Pi_1 = P_1 = \text{orthogonal projection onto } E_1 = S_1$$

$$\Pi_2 = P_0 = \text{orthogonal projection onto } E_0$$

and

$$\Pi = P_1 + P_0 = P_R = \text{orthogonal projection onto } R(LB) = E_1 \oplus E_0 .$$

Moreover,

$$\overline{R(\hat{L}(T)B)} = Z_1 = s_2(E_1) \qquad \text{and} \qquad R(L(T)B)^{\perp} = Z_2 = s_2(E_4) .$$

Now consider the projection $\bar{P}'$ defined by (4.3.8) (see example 4.3.6) and let $P_R$ = orthogonal projection onto $\overline{R(LB)}$ be given by (5.9.21) (see example (iii) of §5.9.2). Here we have

$$\bar{P}' = P_R = \text{orthogonal projection onto } \overline{\mathrm{Span}(R)} = \overline{R(LB)}$$

$$\overline{R(L(\cdot)B)} = F_1 = s_1(E_1) \qquad \text{and} \qquad R(L(\cdot)B)^{\perp} = F_2 = s_1(E_2) .$$

So the topologies of $X$ constructed in examples (ii) and (iii) of §5.9.2 are such that the orthogonal projections onto $R(LB)$ are respectively $\Pi$ and $\bar{P}'$ .

These representations for $U$ and $X$ also allow us to write $(LB)^{\dagger}$ , $(L(\cdot)B)^{\dagger}$ , $(L(T)B)^{\dagger}$ , etc. in simple expressions. For example,

$$(LB)^{\dagger} z = \sum_{i=0,1} \sum_{j \in \Lambda_i} a_{ij} e_j$$

for $z = \left( \sum_{i=0}^{4} \sum_{j \in \Lambda_i} a_{ij} \Phi_{ij} \right) \in X$ .

5.10.2 - Remark:

We shall see in section 6.7 that $R = \{LBe_j\}_{j\in\Delta}$ will be linearly independent in several practical applications where $\{e_j\}_{j\in\Delta}$ is a sequence of linearly independent functions $e_j:[0,T] \to U$ (such as sines, cosines or monomials, etc.). However, in general we can always find a complete matched set

$$\mathcal{M} = (\{e_j\}_{j\in\Gamma} \;,\; \{\Phi_j\}_{j\in\Lambda} \;,\Delta,\Gamma,\Lambda)$$

for the primitive operator of $LB$, by applying the techniques presented in section 5.8 (THE GENERATION OF COMPLETE MATCHED SETS) and, once we have $\mathcal{M}$, the set

$$R = \{\Phi_j\}_{j\in\Delta} = \{LBe_j\}_{j\in\Delta}$$

is linearly independent, as well as $\{e_j\}_{j\in\Delta}$.

5.10.3 - First procedure:

Here we suppose that $\mathcal{U} = H^r(0,T;U)$ for some $r \in \mathbb{R}$ (or $\mathcal{U} = H_0^r(0,T;U)$ for some $r \geq 0$) is fixed and we have the linearly independent set

$$R = \{LBe_j\}_{j\in\Delta}$$

where $e_j:[0,T] \to U$ are linearly independent functions. Also let $\mathcal{X}$ be given by (5.10.6).

In order to have either condition (5.10.4) or (5.10.5) holding it is clear, by (5.10.7), that only the values of $\beta_{0j}$ and $\beta_{1j}$ will matter since $\beta_{ij}$ for $2 \le i \le 4$ will only alter the topology of $R(LB)^{\perp}$ . Theorems 5.9.4 and 5.9.5 show one possible way to select these values of $\beta_{0j}$ and $\beta_{1j}$ . After this we can try to associate $X$ with some $s \in \mathbb{R}$ such that

$$X \simeq M^s(0,T;Z)$$

by using the imbedding corollary 5.7.8. That is, try to find for each value $s$ the space $R(LB) = E_0 \oplus E_1$ is a closed subspace (imbedded) in $M^s(0,T;Z)$ . The larger $r$ is, the larger $s$ becomes, that is the smoother $U$ is the smoother $X$ will become. Actually we do not need to associate $X$ with some $M^s$ , we can work with $X$ itself. For example, after selecting $\beta_{0j}$ and $\beta_{1j}$ as indicated above we can choose any values for $\beta_{2j}$ , $\beta_{3j}$ and $\beta_{4j}$ , e.g., $\beta_{2j} = \beta_{3j} = \beta_{4j} = 1$ or $\beta_{ij} = \|\Phi_{ij}\|^2_{M^s}$ for $2 \le i \le 4$ (in which case $R(LB)^{\perp}$ will be imbedded in $M^s$) and then we have a full description of the space $X$ given by (5.10.6).

So, in the above procedure we set $U = H^r$ (or $H^r_0$) and constructed the spaces $X$ ($\simeq M^s$) which will contain the pair $(z(\cdot),z_f)$ consisting of the trajectory and final state. Let us now see an alternative procedure:

5.10.4 - <u>Second procedure</u>:

Here we let $X = M^s(0,T;Z)$ , for some $s \in \mathbb{R}$ , fixed and assume again that for the linearly independent sequence $\{e_j\}_{j\in\Delta}$ of functions from

$[0,T]$ to $U$ the set

$$R = \{LBe_j\}_{j\in\Delta}$$

is linearly independent. Also set $\mathcal{U}$ as in (5.10.8), that is

$$\mathcal{U} = \{u = \sum_{j\in\Delta} u_j e_j : \quad \| u \|_{\mathcal{U}}^2 = \sum_{j\in\Delta} |u_j|^2 \alpha_j\} \quad .$$

Theorem 5.9.3 shows one possible way to select $\alpha_j$ in order to have condition (5.10.4) or (5.10.5) holding. After this we can try to associate $\mathcal{U}$ with some $r \in \mathbb{R}$ for which

$$\mathcal{U} \simeq H^r(0,T;U)$$

by applying the imbedding corollary 5.7.8. We can also think of $\mathcal{U}$ as being our space of input functions (without having to associate with any $H^r$) . For example, suppose $U = \mathbb{R}$ and $e_j(t) = t^j$ , $j \in \Delta = \mathbb{Z}^+ = \{0,1,2,\ldots\}$. In this case our space of input functions $\mathcal{U}$ will be polynomials of the form

$$u(t) = u_0 + u_1 t + u_2 t^2 + \ldots$$

normalized by

$$\| u \|_{\mathcal{U}}^2 = \sum_{j=0}^{\infty} |u_j|^2 \alpha_j \quad .$$

Also note that: the smoother we set $X = M^s$ , the smoother $\mathcal{U}$ will become.

Observe that it makes no difference if we identify $X$ here with

$X$ in the more convenient form (5.10.6) as long as we set

$$\beta_{ij} = \| \Phi_{ij} \|^2_{M^s} \qquad j \in \Lambda_i \ , \quad i \in \{0,1,2,3,4\}$$

since, by the imbedding corollary 5.7.8, these two spaces have equivalent topology. In the case $i = 0,1$ $\beta_{ij}$ becomes

$$\beta_{ij} = \| L(\cdot)Be_j \|^2_{H^s} + \| L(T)Be_j \|^2_Z \qquad j \in \Lambda_i \ .$$

Whereas $\{e_j / \sqrt{\alpha_j}\}_{j\in\Delta}$ is a c.o.s. in $U$, $\{LBe_j / \sqrt{\beta_j}\}_{j\in\Delta}$ for $\beta_j$ defined by (5.9.22) is a c.o.s. in $X$.

5.10.5 - <u>A note on $R(L(T)B)$ and the state space $Z$</u>.

Let $U$ be given by (5.10.8) again, i.e.,

$$U = U'(e_j, \alpha_j, \Delta)$$

and suppose that the functions in $U$ have values in the input space $U$. Let $R$ be as in example (ii) of §5.9.2, i.e., $R$ is the linearly independent set

$$R = \{LB\ e_j\}_{j\in\Delta} \ .$$

Set

$$V = R(L(T)B) \tag{5.10.9}$$

that is, $V$ is the reachable set from the origin of the system formed by the linear part of (3.2.7) (see section 3.1).

We have seen in theorem 5.9.3 that if

$$\alpha_j \leq \| LBe_j \|^2_{M^s} \qquad \text{for all} \quad j \in \Delta \tag{5.10.10}$$

then $\{\alpha_j / \| LBe_j \|^2_{M^s}\}_{j\in\Delta}$ is bounded and therefore (5.10.4) holds, that is

$$LB \text{ has closed range in } X = M^s(0,T;Z) .$$

In particular, if

$$\alpha_j = \| LBe_j \|^2_{M^s} \qquad \text{for all} \quad j \in \Delta \tag{5.10.11}$$

then both $\{\alpha_j / \| LBe_j \|^2_{M^s}\}_{j\in\Delta}$ and $\{\| LBe_j \|^2_{M^s} / \alpha_j\}_{j\in\Delta}$ are bounded and therefore (5.10.5) holds, that is

$$LB \text{ is bounded and has closed range in } X = M^s(0,T;Z) .$$

So, by manipulating $\alpha_j$ we can give the right topology to $U$ in order to either (5.10.4) or (5.10.5) be satisfied. Also, the smoother $X$ is the smoother (in t) $U$ becomes. (In other words, the larger s is the smoother $U$ becomes.)

It is interesting to note that if $\alpha_j$ is such that LB has closed range in $X = M^s(0,T;Z)$ then, for $Z_1$ and $Z_2$ defined by (5.9.13) we have that

$$V = (Z_1, \| \cdot \|_Z) \qquad \text{and} \qquad V^{\perp} = (Z_2, \| \cdot \|_Z) \tag{5.10.12}$$

That is, the above manipulation of $\alpha_j$ not only make $R(LB)$ closed in $X$ , but also $R(L(T)B)$ closed in $Z$ .

Actually, in cases where $Z$ is infinite dimensional we shall often have

$$V = Z \qquad \text{and} \qquad V^{\perp} = \{0\} . \tag{5.10.13}$$

Also, if we take a smoother state space $Z' \subset Z$ , redefine $X = M^S(0,T;Z')$ then the functions $u$ in $U$ will become smoother with respect to $U$ . That is the functions $u \in U$ will now have values in a smoother space $U' \subset U$ . We shall see this in examples in section 6.7.

We have seen in §4.7.5 that if (5.10.13) holds, then the projections $\Pi_1$ , $\Pi_2$ and $\Pi$ satisfy the following for any $z = (z(\cdot),z_f) \in M^S(0,T;Z)$:

$$\begin{aligned} \Pi_1 z &= \Pi_1(z(\cdot),z_f) = (L(\cdot)Bu',z_f) \\ \Pi_2 z &= \Pi_2(z(\cdot),z_f) = (L(\cdot)Bu'',0) \end{aligned} \tag{5.10.14}$$

and

$$\Pi z = \Pi\,(z(\cdot),z_f) = (L(\cdot)Bu,z_f)$$

for some $u', u'', u \in U$ such that $u = u' + u''$ ,

$$L(T)Bu' = z_f \qquad \text{and} \qquad L(T)Bu'' = 0 .$$

5.10.6 - <u>Examples of projections onto $S = LB(U_{ad})$</u>:

Let us first take $U_{ad}$ as in (5.10.1), i.e., $U_{ad} = U$ . Then

$$S = R(LB) .$$

Once LB satisfies (5.9.6), (i.e., $\mathcal{R}(LB)$ is closed), we have that $\Pi$ and $\bar{P}'$ are two examples of continuous linear projections onto S . If $V = Z$ (i.e., if $\mathcal{R}(L(T)B) = Z$) then $\Pi$ is also uniform in the first component. Other examples of uniform projections are given by $P' = \tilde{P}$ given by (4.7.6) and $P' = \tilde{P}'$ given by (4.7.7), that is

$$P'z = \bar{P}'(z) \;/\; \phi_S(z) \tag{5.10.15}$$

and

$$P'z = \bar{P}'(\phi_S(z).z) \; . \tag{5.10.16}$$

However, unlike $\Pi$ , $P'$ in either (5.10.15) or (5.10.16) is not linear.

Now take $\mathcal{U}_{ad} \subset \mathcal{U}$ as in (5.10.2), i.e.,

$\mathcal{U}_{ad}$ is bounded, closed and convex.

We have seen that if LB satisfies (5.10.4) (i.e., if LB is bounded and $\mathcal{R}(LB)$ is closed) then, by theorem 5.2.4, $S = LB(\mathcal{U}_{ad})$ satisfies

S is bounded, closed and convex.

One natural choice of a continuous projection $P$ onto S of this type would certainly be the projection (4.1.3), namely

$Pz$ = the closest element to z in S .

Now observe that $S \subset \mathcal{R}(LB)$ and therefore we can find $P':\mathcal{R}(LB) \to \mathcal{R}(LB)$ a continuous projection onto S . Using the projection defined by (4.1.13) (see example (iii) of §4.1.8), we have that

$$\mathcal{P} = P'\Pi \qquad \text{and} \qquad \mathcal{P} = P'\bar{P}'$$

are two continuous projections onto $S$ . Moreover, if $P_{ad}:\mathcal{U} \to \mathcal{U}$ is a continuous projection onto $\mathcal{U}_{ad}$ , then

$$\mathcal{P} = (LB)P_{ad}(LB)^{\dagger}$$

is also a continuous projection onto $S$ .

# CHAPTER 6.

# NONLINEAR CONTROLLABILITY.

## 6.1 - INTRODUCTION.

### 6.1.1 - The problem of nonlinear controllability.

The problem of nonlinear controllability which we study here is: for a given desired state $z_d \in Z$ and a given $\varepsilon \geq 0$ we want to determine whether there is a control $u^* \in \mathcal{U}_{ad}$ which drives the system from the initial state $z_0$ at $t = 0$ to a final state $z_f \in \overline{B_\varepsilon(z_d)}$ at $t = T$. In the terminology of chapter 3 this is $\varepsilon$-controllability to $z_d$. In particular, since we admit the case of $\varepsilon$ being null, the problem is equivalent to exact controllability when we set $\varepsilon = 0$. We call $u^*$ a "wanted control".

Consider the system described by (3.2.7), namely

$$\begin{aligned} z &= Sz_0 + LBu + LNz(\cdot) \\ z(0) &= z_0 \quad \text{and} \quad z(T) = z_f \end{aligned} \tag{6.1.1}$$

where $z = (z(\cdot), z_f) \in X$ = a space of the type $M(0,T;Z)$ (see §2.2.6). We shall first concentrate here mainly in two cases for the set of admissible controls $\mathcal{U}_{ad}$.

$$\mathcal{U}_{ad} \text{ is bounded, closed and convex set in } \mathcal{U} \tag{6.1.2}$$

and

$$\mathcal{U}_{ad} = \mathcal{U} \quad \text{or} \quad \mathcal{U}_{ad} = \text{closed subspace of } \mathcal{U} . \tag{6.1.3}$$

In §6.4.10 we consider more general types of admissible controls. For the case (6.1.2) we assume that (5.10.5) holds (i.e., $LB \in \mathcal{L}(\mathcal{U},X)$ and has closed range in $X$) and for the case (6.1.3) we assume that the weaker condition (5.10.4) holds (i.e., $R(LB)$ is closed in $X$). We have seen in section 5.10 how to adjust the spaces $\mathcal{U}$ and $X$ in order to have these conditions. More specific examples will be shown in section 6.7. Define

$$S = LB(\mathcal{U}_{ad}) .$$

Clearly, by definition, system (6.1.1) is $\varepsilon$-controllable to $z_d$ if and only if

$$z - Sz_0 - LNz(\cdot) \in S \tag{6.1.4}$$

and

$$z_f \in \overline{B_\varepsilon(z_d)} \tag{6.1.5}$$

for some $z \in X$ .

This statement would not be true if we substituted (6.1.4) by

$$z(\cdot) - S(\cdot)z_0 - L(\cdot)Nz(\cdot) \in L(\cdot)B(\mathcal{U}_{ad})$$

since in this case we would not necessarily have $z_f = z(T)$ . That is, in the present approach we must look at the pair $(z(\cdot),z_f)$ trajectory-final state in a space $X$ of the type $M(0,T;Z)$ rather than the trajectory $z(\cdot)$ only.

In the particular case of exact controllability to $z_d$, i.e., $\varepsilon = 0$, (6.1.5) becomes

$$z_f = z_d \quad . \tag{6.1.6}$$

We denote by $P$ and $P_d$ the mappings $P:X \to X$ and $P_d:Z \to Z$

$P$ = any continuous projection onto $S$, and

$P_d$ = any continuous projection onto $\overline{B_\varepsilon(z_d)}$.

The simplest example of $P_d$ is given by (4.1.18), the translated $\varepsilon$-radial retraction. Examples of continuous projections $P$ onto $S = LB(U_{ad})$ are given in §5.10.6, for both cases $U_{ad}$ as in (6.1.2) and $U_{ad}$ as in (6.1.3).

We also define $\xi:X \to X$ by

$$\xi(x) = x - Sz_0 - LNx(\cdot) \ , \quad x = (x(\cdot),x_f) \in X \tag{6.1.7}$$

and hence we can write (6.1.4) as

$$\xi(z) \in S \ . \tag{6.1.8}$$

Now observe that $z = z^*$ satisfies (6.1.5) and (6.1.8) if and only if there is a wanted control $u^*$ given by

$$u^* = (LB)^{\dagger}\xi(z^*) + u'$$

for some $u' \in N(LB)$ . We shall call $z^*$ a "wanted state". It is clear that to obtain one wanted control it is enough to have one wanted state and if we have the set of wanted states then we can determine the set of wanted controls. We denote

$$W_s^{\varepsilon} = \text{the set of all wanted states.}$$

In the particular case of exact controllability, $\varepsilon = 0$ , the set of wanted states is then represented by $W_s^0$ .

6.1.2 - The mappings $F$ .

In sections 6.2 and 6.4 we present some mappings $F: X \to X$ and information about the fixed points of $F$ will give us knowledge about the wanted states of the system. Each mapping $F$ will have an operative function $f: X \to X$ and we also denote $\nu(F)$

$$\nu(F) = \text{the set of fixed points of } F .$$

If elements of $\nu(F)$ can be obtained then elements of $W_s^{\varepsilon}$ may be obtained from the set

$$f(\nu(F)) .$$

In most of the cases $f = I$ , the identity on $X$, so that we shall be able to obtain wanted states directly from the fixed points of $F$ .

The ideal mapping $F$ would be such that

$$W_s^\varepsilon = f(\nu(F)) ,$$

that is

$$x \in f(\nu(F)) \iff x \in W_s^\varepsilon .$$

In this case the mapping $F$ provides necessary and sufficient conditions for us to obtain the wanted states from its fixed points. Unfortunately this will not always be the case. We say that $F$ has necessary conditions (to obtain the wanted states) if $f(\nu(F)) \subset W_s^\varepsilon$ , which is equivalent to

$$x \in f(\nu(F)) \Rightarrow x \in W_s^\varepsilon ,$$

and we say that $F$ has sufficient conditions (to obtain the wanted states) if $f(\nu(F)) \supset W_s^\varepsilon$ , which is the same as

$$x \in f(\nu(F)) \Leftarrow x \in W_s^\varepsilon .$$

We shall see in the next section that some earlier attempts to use fixed points in nonlinear controllability used mappings $\phi$ which in some cases did not provide neither necessary nor sufficient conditions. Generally speaking mappings $F$ with necessary conditions are more difficult to be obtained than mappings $F$ with sufficient conditions so that we shall sometimes consider acceptable mappings $F$ which do not have necessary conditions. In this case however, a test will be given to check each element of $f(\nu(F))$ whether it is a wanted state or not. We refer to this test as the "a posteriori test".†

Consider the following example of $F$ :

$$F(x) = \mathcal{S}z_0 + LNx(\cdot) + \mathcal{P}\xi(x) \tag{6.1.9}$$

and set $f = I$ . Clearly, since $\mathcal{P}\xi(x) = \xi(x)$ , for $\xi(x) \in S$ , by (6.1.8), any wanted state is a fixed point of $F$ and therefore $F$ has sufficient conditions. However, $F$ do not have necessary condition since each fixed point $(x(\cdot),x_f)$ of $F$ will have to satisfy (6.1.5) to be a wanted state. Condition (6.1.5) can be regarded as the posteriori test in this case.

So, we are interested in the set $f(\nu(F)) \cap W_S^\varepsilon$ , the elements of the operative set $f(\nu(F))$ which satisfy the a posteriori test. The set

$$J_S(F) = W_S^\varepsilon \setminus f(\nu(F)) \tag{6.1.10}$$

represents all the wanted states which cannot be achieved by the fixed points of $F$ . Clearly if $F$ has sufficient conditions $J_S(F)$ is empty. Similarly the set

$$J_N(F) = f(\nu(F)) \setminus W_S^\varepsilon \tag{6.1.11}$$

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† The author first used this name in [15] since it was meant to test each fixed point of $F$ after $\nu(F)$ had been obtained.

represents the elements of $f(\nu(F))$ which fail the a posteriori test and if $F$ has necessary conditions $J_N(F) = \emptyset$. When sufficiency is not possible for a particular mapping $F$, then we would like that $J_N(F)$ does not have many elements. For $F$ in (6.1.9) there may be many elements in $J_N(F)$, since every mild solution of the dynamical equation is a fixed point of $F$, and therefore we do not consider it suitable for our purposes. We shall pursue mappings $F$ with a more restricted set $J_N(F)$.

Every mapping $F$ presented here will be continuous. This is essential for both the proof of existence of a fixed point as well as in the search for a fixed point of $F$.

### 6.1.3 - Desired properties of $F$.

The most obvious properties we want $F$ to have are:

(a) $F$ has necessary conditions.

(b) $F$ has sufficient conditions.

As mentioned in §6.1.2, if (a) does not hold there will be an a posteriori test provided. It is important to note that the big disadvantage of a mapping $F$ which does not satisfy (a) is not the fact that a test will have to be applied to the elements of $f(\nu(F))$ but the fact that we cannot guarantee existence of wanted states (and therefore existence of wanted controls) by existence of fixed points of $F$ (which may be determined by using the fixed point theorems we analysed in section 2.6).

It is also desirable that F includes our approach of $\varepsilon$-controllability rather than just exact controllability to $z_d$, so that we want F to satisfy:

(c) F is parametrized by $\varepsilon$ , $\varepsilon \geq 0$ .

We want to consider not only the case $\mathcal{U}_{ad} = \mathcal{U}$ but also the case $\mathcal{U}_{ad}$ as in (6.1.2). Therefore, the property

(d) F admits $\mathcal{U}_{ad}$ in both cases (6.1.2) and (6.1.3)

is also desirable.

Finally, observe that $\mathcal{R}(L(T)B)$ is a closed subspace of Z (see §5.10.5), and the system formed by the linear part of (6.1.1) is

$$z(\cdot) = \mathcal{S}(\cdot)z_0 + L(\cdot)Bu \ , \qquad z(0) = z_0 \ . \tag{6.1.10}$$

We have seen in §3.1.1 that the reachable set of system (6.1.10) is contained in the set

$$\mathcal{S}(T)z_0 + \mathcal{R}(L(T)B)$$

and this implies that, unless $\mathcal{U}_{ad} = \mathcal{U}$ and $\mathcal{R}(L(T)B) = Z$ , we cannot control the linear system (6.1.10) to the whole state space Z . However, for the nonlinear system (6.1.1) there is no reason for us to think, a priori, that if

$$\mathcal{R}(L(T)B \neq Z$$

(i.e., if $\mathcal{R}(L(T)B)$ is a proper subspace of $Z$) we cannot control the system to a particular desired state $z_d \notin S(T)z_0 + L(T)Bu$. So, when developing the mappings $F$ we should bear in mind the possibility

(e) $F$ admits cases where $\mathcal{R}(L(T)B) \neq Z$.

Only few mappings which we obtain here will satisfy all the five properties (a)-(e).

## 6.2 - HISTORICAL VIEW OF PREVIOUS WORK.

Fixed point theorems have been used since the beginning of the century to show local existence theorems for differential equations. In nonlinear controllability the relevant papers which treated the problem by an approach via fixed points were [25,9,35,15]. In order to draw comparisons we write the results of each paper using our notation. In this section $f = I$ so that the operative set

$$f(\nu(F)) = \nu(F) .$$

First we observe that these papers only considered the cases

$$\mathcal{U}_{ad} = \mathcal{U} \quad \text{and} \quad \varepsilon = 0 .$$

That is, the cases $\mathcal{U}_{ad} \neq \mathcal{U}$ and $\varepsilon$-controllability are only being introduced in the present thesis. Also, the approach using the pair trajectory-final state was only introduced in [15] by the author. The first three papers [25,9, and 35] employed a map $\phi$ to obtain the component $z^*(\cdot)$ (i.e., the trajectory of the system) of the wanted state $(z^*(\cdot), z_f^*)$ .

Both [25] and [9] utilized the mapping

$$\phi z(\cdot) = L(\cdot)B\tilde{G}^{-1}(z_d - L(T)Nz(\cdot)) + L(\cdot)Nz(\cdot) \tag{6.2.1}$$

where $\tilde{G}: \mathcal{U}/\mathcal{N}(G) \to Z$ was defined by $\tilde{G}[u] = Gu$ for all equivalent classes $[u] \in \mathcal{U}/\mathcal{N}(G)$ and $G$ is given by

$$G = L(T)B \ .$$

It was assumed that $z_0 = 0$ and $\mathcal{R}(G) = \mathcal{R}(L(T)B)$ satisfied

$$\mathcal{R}(L(T)B) = Z \tag{6.2.2}$$

so that the inverse $\tilde{G}^{-1}$ of $\tilde{G}$ always existed. Clearly for a fixed point $z^*(\cdot)$ of $\phi$ (i.e., for $z^*(\cdot) \in \nu(\phi)$) , we have that $z^*(\cdot) = \phi z^*(\cdot)$ and if we set

$$z^* = (z^*(\cdot), z^*(T)) \tag{6.2.3}$$

$z^*$ satisfies (6.1.4) and (6.1.6) and hence is a wanted state (i.e., $z^* \in W_s^0$) . So, $\phi$ in (6.2.1) has necessary conditions, in the sense of §6.1.2, for the case $z_0 = 0$ , $\mathcal{U}_{ad} = \mathcal{U}$ and $\varepsilon = 0$ . Now observe that

$$(\phi z(\cdot) - LNz(\cdot),(\phi z(\cdot))(T)-L(T)Nz(\cdot)) \in S_1 \quad \forall\, z(\cdot) \qquad (6.2.4)$$

where $S_1$ is given by (4.3.10). But $S_1$ is strictly contained in $S$ (since $\mathcal{N}(L(T)B)^{\perp}$ is strictly contained in $\mathcal{N}(L(\cdot)B)^{\perp} = \mathcal{N}(LB)^{\perp}$) and hence there may be a wanted state $z^* \in S\backslash S_1$ . By (6.2.4) $z^* \notin \mathcal{R}(\phi)$ and this implies that $z^* \notin \nu(\phi)$ . In other words, $\phi$ does not have sufficient conditions. Summarizing, $\phi$ in (6.2.1) satisfies (a) for the case $z_0 = 0$ , $\mathcal{U}_{ad} = \mathcal{U}$ and $\varepsilon = 0$ but it does not satisfy any of the conditions (b)-(e).

In [35] A.J. Pritchard used the mapping

$$\phi z(\cdot) = L(\cdot)BG^{\dagger}(z_d - L(T)Nz(\cdot)) + L(\cdot)Nz(\cdot) . \qquad (6.2.5)$$

The assumption (6.2.2) could be eliminated, that is $\phi$ in (6.2.5) satisfies (e). Instead, the following condition was introduced:

$$\mathcal{R}(L(T)B) \text{ is closed in } Z . \qquad (6.2.6)$$

It was assumed in [35] that $\mathcal{U}$ and/or $Z$ could be adjusted in order to the operator $G = L(T)B$ have closed range in $Z$ so that (6.2.6) is satisfied and $G^{\dagger}$ is continuous. We mention at this point that the author developed the results of chapter 5 after examining deeply the possibility of this adjustment being feasible.

This improvement had a cost, the necessity that $\phi$ in (6.2.1) satisfied was lost and the following a posteriori test had to be satisfied by $z^*(\cdot) \in \nu(\phi)$

$$[z_d - L(T)Nz^*(\cdot)] \in \mathcal{R}(L(T)B) \quad . \tag{6.2.7}$$

(Note: condition (6.2.7) was called in [35] "check of consistency" for each fixed point of $\phi$). Now observe that (6.2.4) also holds for $\phi$ in (6.2.5) and therefore it does not have sufficient conditions either. Summarizing, $\phi$ in (6.2.5) satisfies (e) but it does not satisfy (a)-(d).

In [15] the author improved the above results by introducing linear projections and the space $X = M^2(0,T;Z)$ . Still in the case $\mathcal{U}_{ad} = \mathcal{U}$ and $\varepsilon = 0$ (but admitting $z_0 \neq 0$), we presented in [15] the class of mappings $F:X \to X$ (parametrized by the projection $P$ used) which not only eliminated the assumption (6.2.2) but also provided sufficient conditions. In other words F satisfied (b) and (e). These mapping F had the form

$$F(z) = \zeta + (I-P)LNz(\cdot) + Pz - LBG^{\dagger}z_f \tag{6.2.8}$$

where $\zeta \in X$ is a fixed element in $X$ given by

$$\zeta = LBG^{\dagger}z_d + (I-P)Sz_0 \ .$$

Similarly to assumption (6.2.6) in [35] it was assumed that $\mathcal{R}(LB)$ is closed and that this could be achieved by adjusting $\mathcal{U}$ and/or Z . However, as in $\phi$ of (6.2.5), the necessity does not hold for F in general. The following a posteriori test was given in [15]

$$[z_d - L(T)Nz^*(\cdot) - S(T)z_0] \in \mathcal{R}(L(T)B) \tag{6.2.9}$$

which obviously is the same as the check of consistency in [35] for cases where $z_0 = 0$ . It was also shown in [15] and it is easy to verify that when $z_0 = 0$ and $P$ is the projection $\Pi = \Pi_1 + \Pi_2$ of (4.3.12), then

$$F(z) = (\phi z(\cdot),(\phi z(\cdot))(T)) + \Pi_2(z-LNz(\cdot)) \qquad (6.2.10)$$

where $\phi$ is as in (6.2.5). That is, $\phi$ is the particular case of this approach when $P = \Pi$ . (The second summand of the LHS of (6.2.10) could be interpreted as the missing term in (6.2.5) which gives sufficiency.) Moreover, $z^*(\cdot) \in \nu(\phi) \Rightarrow (z^*(\cdot),z^*(T)) \in \nu(F)$ whereas the converse is not true. So, in general F satisfied (b) and (e) but it does not satisfy (a), (c) and (d).

The author has also shown in [15] that if (6.2.9) holds for all $z(\cdot) \in L^2(0,T;Z)$ , then F provides necessary and sufficient conditions. This included the cases where (6.2.2) holds (i.e., F generalizes $\phi$ in (6.2.1) too, when the same assumptions were imposed) and also cases such as for example if $z_0 = 0$ , the desired state $z_d$ lies in $R(L(T)B)$ and $L(T)Nz(\cdot) \in R(L(T)B)$ for all $z(\cdot) \in L^2(0,T;Z)$ .

## 6.3 - THE ANALYSIS OF $\phi$ AND F .

Here we introduce a theorem which will help in the analysis of $\phi$ and F of last section as well as to provide other improved mappings F .

6.3.1 - Theorem.

Let $X$ be a normed vector space, $S'$ and $S''$ be two subsets of $X$, $P:X \to X$ be a projection onto $S'$, $\eta:X \to X$ be any mapping, $Q:X \to X$ be any active mapping under $P$ and $F:X \to X$ be given by

$$F(x) = x - \eta(x) + P\eta(x) - Q(x) . \tag{6.3.1}$$

i) If $S' \subset S''$, then we have

$$x^* \in \nu(F) \Rightarrow \begin{cases} Q(x^*) = 0 \\ \eta(x^*) \in S' \end{cases} \tag{6.3.2}$$

ii) If $S' \supset S''$, then we have

$$x^* \in \nu(F) \Leftarrow \begin{cases} Q(x^*) = 0 \\ \eta(x^*) \in S'' . \end{cases} \tag{6.3.3}$$

iii) If $S' = S''$, then we have

$$x^* \in \nu(F) \Leftrightarrow \begin{cases} Q(x^*) = 0 \\ \eta(x^*) \in S'' . \end{cases} \tag{6.3.4}$$

Proof:

Clearly $x^* \in \nu(F)$ if and only if

$$\eta(x^*) - P\eta(x^*) + Q(x^*) = 0 . \tag{6.3.5}$$

Let $S' \subset S''$ then, since $Q$ is active under $P$, (6.3.5) implies that

$$Q(x^*) = 0$$

and

$$(I-P)\eta(x^*) = 0 \; . \tag{6.3.6}$$

By property 4.1.2, (6.3.6) is equivalent to

$$\eta(x^*) \in S' \subset S''$$

and therefore (6.3.2) holds. This proves (i).

Now let $S' \supset S''$ then, for $\eta(x^*) \in S''$ , $P\eta(x^*) = \eta(x^*)$ and if $Q(x^*) = 0$ (6.3.1) becomes

$$F(x^*) = x^* \; .$$

This proves (ii). Clearly (i) and (ii) => (iii) .

Q.E.D.

6.3.2 - <u>Remark.</u>

If $Q = 0$ then, since $0$ is active under any projection (see §4.5.3), the theorem holds with $F$ given by

$$F(x) = x - \eta(x) + P\eta(x) \; . \tag{6.3.7}$$

We shall only apply theorem 6.3.1 with $P$, $Q$ and $\eta$ continuous, so that $F$ will be continuous. Often we make $Q = 0$ and use the mapping $F$ in (6.3.7). This is because active mappings $Q \neq 0$ might not exist in cases where $S$ is bounded (see §4.5.4).

6.3.3 - <u>The analysis of $\phi$</u> .

Let $\mathcal{U}_{ad} = \mathcal{U}$, $\varepsilon = 0$ , $z_0 = 0$ , $\xi : X \to X$ given by (6.1.7),

$\Pi_1 : X \to X$ be given by (4.3.9), $S_1$ given by (4.3.10) and

$$S'' = \mathcal{R}(LB) .$$

For $\phi$ in (6.2.5) define $\Phi : X \to X$ by

$$\Phi(z) = (\phi z(\cdot), (\phi z(\cdot))(T)) .$$

It is easy to verify that $\Phi$ can be expressed in the form of $F$ in (6.3.1) with

$$\eta = \xi \ , \quad P = \Pi_1 \ , \quad S' = S_1 \quad \text{and}$$

$$Q(z) = \Pi_1(z(\cdot), z_d - z_f) . \tag{6.3.8}$$

Clearly $Q$ is active under $\Pi_1$ since both $Q$ and $\Pi_1$ are linear and $\mathcal{R}(Q) \subset \mathcal{R}(\Pi_1)$ (see section 4.5).

Applying theorem 6.3.1, since $S_1$ is strictly contained in $S''$, we obtain that (6.3.3) does not hold (which implies that $\Phi$ does not have sufficiency), but (6.3.2) does, that is

$$z^* = \Phi(z^*) \Rightarrow \begin{cases} Q(z^*) = 0 \\ \xi(z^*) \in S . \end{cases} \tag{6.3.9}$$

Therefore $\Phi$ does not have necessary condition either (since $Q(z^*) = 0$ is not equivalent to (6.1.6)), but it is easy to see that the condition

$$Q(z^*) = \Pi_1(z^*(\cdot), z_d - z_f^*) = 0$$

together with the a posteriori test (6.2.7) do imply (6.1.6).

The present analysis of the mapping $\phi$ is different to the analysis made by the author in [15] and also much shorter, thanks to theorem 6.3.1 introduced here.

6.3.4 - The analysis of F .

Let us now take F in (6.2.8). Thanks to theorem 6.3.1 again the present analysis of F is shorter than the way F was first introduced by the author in [15]. Let $\mathcal{U}_{ad} = \mathcal{U}$ , $\varepsilon = 0$ , $\xi : X \to X$ be given by (6.1.7) and

$$S = \mathcal{R}(LB) .$$

Clearly F can be expressed in the form (6.3.1) with

$$\eta = \xi \quad , \quad P = \mathcal{P} \quad , \quad S' = S$$

and Q as in (6.3.8) again, that is

$$F(z) = \mathcal{S}z_0 + LNz(\cdot) + \mathcal{P}\xi(z) - Q(z) . \qquad (6.3.10)$$

Applying theorem 6.3.1, since Q is active under $\mathcal{P}$ , we have that (6.3.4) holds which clearly implies that F has sufficient conditions and

$$z^* = F(z^*) \Rightarrow \begin{cases} Q(z^*) = 0 \\ \xi(z^*) \in S \end{cases}$$

which is similar to (6.3.9) for $\Phi$ . Therefore F does not have necessary conditions and the a posteriori test (6.2.9) had to be introduced.

In the next section we apply theorem 6.3.1 to obtain new classes of mappings F .

## 6.4 - NEW CLASSES OF MAPPINGS F .

Here we create new classes of mappings F and at the same time try to implement properties (a)-(e). Theorems 6.3.1 (from last section) and theorem 6.4.6 (introduced here) will play an important role.

We recall, from §6.1.1 that system (6.1.1) is $\varepsilon$-controllable to $z_d$ if and only if for $z \in X$ , called wanted state, we have

$$\xi(z) \in S = LB(\mathcal{U}_{ad}) \tag{6.4.1}$$

and

$$z_f \in \overline{B_\varepsilon(z_d)} \quad \text{in} \quad Z \tag{6.4.2}$$

where $\xi: X \to X$ is defined for $z = (z(\cdot), z_f) \in X$ by (6.1.7), namely

$$\xi(z) = z - Sz_0 - LNz(\cdot) .$$

Also, $P$ and $P_d$ represent any continuous projections onto S and $\overline{B_\varepsilon(z_d)}$ respectively. If $\varepsilon = 0$ , i.e., the case of exact controllability, (6.4.2) becomes

$$z_f = z_d . \tag{6.4.3}$$

6.4.1 - Properties (a), (b), (c) and (e) satisfied.

We have seen in §6.3.4 that the mapping $F$ in (6.3.10) (with operative function $f = I$), satisfies (b) and (e). By theorem 6.3.1, an immediate generalization can be obtained if we replace $Q$ in (6.3.8) by another $Q$ which is active under $P$ and satisfies

$$Q(z) = 0 \iff z_f \in \overline{B_\varepsilon(z_d)} \quad . \tag{6.4.4}$$

Clearly this would give $F$ necessary conditions, i.e. property (a), and property (c) as well. Also, with (a) holding there is no need for a posteriori test.

If $\mathcal{U}_{ad} = \mathcal{U}$ then we can easily obtain active mappings satisfying (6.4.4). For example, let $Q{:}X \to X$ be a mapping of the type (2.2.4), namely

$$Q(z) = -q(z_f)\,\overline{x} \quad , \quad z = (z(\cdot), z_f) \in X$$

where $\overline{x} \neq 0$, $\overline{x} \in S = R(LB)$ is chosen arbitrarily and $q{:}Z \to \mathbb{R}$ is any functional. Then $Q$ is active under any projection $P$ onto $S$ (see example 4.5.2). Now, if the functional $q$ satisfies

$$q(z_f) = 0 \iff z_f \in \overline{B_\varepsilon(z_d)} \tag{6.4.5}$$

then (6.4.4) holds. We have seen in example (2.1.1) that $q$ given by

(2.1.2)-(2.1.4) are continuous functionals satisfying (6.4.5). Other examples could be easily obtained using projections. For example, by property 4.1.2, the following functionals $q$ are continuous and satisfy (6.4.5):

$$q(z_f) = \|P_d z_f - z_f\|_Z$$

and

$$q(z_f) = \frac{m\|P_d z_f - z_f\|_Z}{1 + \|P_d z_f - z_f\|_Z} \quad , \quad m \neq 0 \, . \tag{6.4.6}$$

For $q$ in (6.4.6) we also have that

$$0 \leq |q(z_f)| < |m| \quad . \tag{6.4.7}$$

Also, in the case $\varepsilon = 0$, exact controllability to $z_d$, the functionals $q$ have obvious particularizations, e.g. (6.4.6) becomes

$$q(z_f) = \frac{m\|z_d - z_f\|_Z}{1 + \|z_d - z_f\|_Z} \quad .$$

Summarizing, $F: X \to X$

$$F(z) = Sz_0 + LNz(\cdot) + P\xi(z) + q(z_f)\overline{x} \tag{6.4.8}$$

satisfies (a), (b), (c) and (e) with operative function $f = I$. Moreover, the operative set $f(\nu(F)) = \nu(F) = W_s^\varepsilon$ and

$$(z(\cdot),z_f) \mapsto q(z_f)\overline{x}$$

is a completely continuous mapping and this may help when applying fixed point theorems.

We do not implement property (d), i.e., $\mathcal{U}_{ad} \neq \mathcal{U}$, with this approach since it may be difficult (or even impossible) to obtain active mappings Q under $\mathcal{P}$ when S is not a subspace of $X$ (see §4.5.4).

6.4.2 - Properties (a), (b), (d) and (e) satisfied.

Now we want to consider the case $\mathcal{U}_{ad} \neq \mathcal{U}$ too, that is, property (d). So here $\mathcal{U}_{ad}$ satisfies either (6.1.2) or (6.1.3). Let us, for the moment, abandon property (c) and consider $\varepsilon = 0$, exact controllability. Let

$$Q = 0 \ , \quad P = \mathcal{P} \ ,$$

and

$$\eta(z(\cdot),z_f) = \xi(z(\cdot),z_d) \qquad \forall\ (z(\cdot),z_f) \in X$$

then F given by (6.3.1) becomes

$$F(z) = (0,z_1-z_d) + \mathcal{S}z_0 + LNz(\cdot) + \mathcal{P}\xi(z(\cdot),z_d) \tag{6.4.9}$$

for $z = (z(\cdot),z_1) \in X$. Now applying theorem 6.3.1 we obtain

$$(z^*(\cdot),z_1^*) \in \nu(F) \iff \xi(z^*(\cdot),z_d) \in S \ .$$

Setting the operative function $f: X \to X$

$$f(z(\cdot), z_1) = (z(\cdot), z_d)$$

we have that

$$f(\nu(F)) = W_s^0$$

and $F$ of (6.4.9) satisfies all properties (a)-(e), except property (c) (since $\varepsilon = 0$).

Note that we only use the first component of the fixed point. In fact, $F$ in (6.4.9) has the property that for any $z_1, z_2 \in Z$,

$$(z^*(\cdot), z_1) \in \nu(F) \Rightarrow (z^*(\cdot), z_2) \in \nu(F) \quad .$$

This follows since the last three summands of the RHS of (6.4.9) do not depend on the component $z_1$. This feature may help in the search for a fixed point of $F$.

Summarizing, $F$ in (6.4.9) satisfies properties (a), (b), (d) and (e).

6.4.3 - All properties (a)-(e) satisfied.

Let

$$Q = 0 \ , \quad P = \mathcal{P}$$

and

$$\eta(z(\cdot), z_f) = \xi(z(\cdot), P_d z_f) \quad , \quad \forall\, (z(\cdot), z_f) \in X$$

then, $F$ in (6.3.1) becomes

$$F(z) = (0,\ z_1 - P_d z_1) + \mathcal{S} z_0 + LNz(\cdot) + \mathcal{P}\xi(z(\cdot), P_d z_1) \qquad (6.4.10)$$

for $z = (z(\cdot), z_1) \in X$ . Applying theorem 6.3.1 we obtain

$$(z^*(\cdot), z_1^*) \in \nu(F) \iff \xi(z^*(\cdot), z_f^*) \in S$$

where $z_f^*$ represents

$$z_f^* = P_d z_1^* \quad .$$

Clearly we have

$$z_f^* \in \overline{B_\varepsilon(z_d)}$$

and therefore, setting the operative function $f: X \to X$

$$f(z(\cdot), z_1) = (z(\cdot), P_d z_1) \qquad (6.4.11)$$

we have that the operative set

$$f(\nu(F)) = W_S^\varepsilon$$

and $F$ in (6.4.10) satisfies all properties (a)-(e).

6.4.4 - <u>Properties (b)-(e) satisfied.</u>

Here we consider an improvement in another direction. Suppose we want to eliminate the first term of the RHS of (6.4.10) so that the mapping $F$ would have the simpler form

$$F(z) = Sz_0 + LNz(\cdot) + P\xi(z(\cdot),P_d z_f) \,. \tag{6.4.12}$$

Observe that we can rewrite (6.4.12) as

$$F(z) = (0,P_d z_f - z_f) + z - \xi(z(\cdot),P_d z_f) + P\xi(z(\cdot),P_d z_f) \,.$$

Clearly if $z_f \in \overline{B_\varepsilon(z_d)}$, $P_d z_f - z_f = 0$ and applying theorem 6.3.1 we get (by (6.3.3) with $Q = 0$)

$$z^* \in \nu(F) \;\; <= \begin{cases} \xi(z(\cdot),P_d z_f) \in S \\ z_f \in \overline{B_\varepsilon(z_d)} \end{cases} .$$

So, setting the operative function $f = I$,

$$z^* \in \nu(F) = f(\nu(F)) <= z^* \in W_s^\varepsilon \tag{6.4.13}$$

which is property (b), sufficient condition. Clearly $F$ also satisfies (c), (d) and (e). However, the simpler form of $F$ in (6.4.12) costs the loss of property (a), necessary conditions, since now the converse of (6.4.13) will not hold in general. Suppose however that for $z^* = (z^*(\cdot),z_f^*) \in f(\nu(F))$, (6.4.2) holds, that is

$$z_f^* \in \overline{B_\varepsilon(z_d)} \tag{6.4.14}$$

then we have that

$$z^* \in f(\nu(F)) \Rightarrow z^* \in W_s^\varepsilon \tag{6.4.15}$$

and therefore (6.4.14) can be regarded as an a posteriori test in this case.

Observe that, in order to be a fixed point, a mild solution of the dynamical equation, say $z^* = (z^*(\cdot), z_f^*)$, has to be such that $z^* - Sz_0 - LNz^*(\cdot)$ is the action of $P$ on $\xi(z^*(\cdot), P_d z_f^*)$, i.e.,

$$z^* - Sz_0 - LNz^*(\cdot) = P\xi(z^*(\cdot), P_d z_f^*) \ .$$

This includes all the wanted states and if $z^* \notin W_s^\varepsilon$, then $z^*$ is eliminated by the a posteriori test. So, unlike F in (6.1.9) where any mild solution is a fixed point of F, here only mild solutions which happen to satisfy the above condition is a fixed point of F. It is clear that the set $J_N(F)$ defined in (6.1.11) is expected to be much smaller than in the case of the mapping F in (6.1.9) and therefore we consider F in (6.4.12) acceptable in spite of having the a posteriori test (6.4.14) to be satisfied.

Summarizing F in (6.4.12) satisfies properties (b)-(e).

### 6.4.5 - Properties (a), (b), (c) and (e) satisfied using uniform projections.

So far we have been free to use any continuous projection $P$ onto S. Now suppose that $U_{ad} = U$ and $P' : X \to X$ is a continuous projection onto $S = R(LB)$ which satisfies

$\mathcal{P}'$ is uniform in the first component

that is, (see §4.7.1) $\mathcal{P}'$ is a continuous projection onto $S$ and

$$\mathcal{P}'(x(\cdot),x_f) = (x(\cdot),\tilde{x}_f) \Rightarrow x_f = \tilde{x}_f \ .$$

We have seen in section 4.7 that $\mathcal{P}' = \Pi$ given by (4.3.12) and also $\mathcal{P}'$ given by (5.10.12) or (5.10.13) are some examples of continuous uniform projections in the first component onto $S$ .

We shall not implement property (d) in this approach since uniform projections are difficult (and sometimes impossible) to be obtained when $S$ is bounded (see §4.7.8). However, all the other properties will hold.

Let $F:X \to X$ as in (6.4.12) using $\mathcal{P} = \mathcal{P}'$ , that is

$$F(z) = \mathcal{S}z_0 + LNz(\cdot) + \mathcal{P}'\xi(z(\cdot),P_d z_f) \tag{6.4.16}$$

and set the operative function $f = I$ again. Now observe, if $z^* = (z^*(\cdot),z_f^*) \in W_S^\varepsilon \Rightarrow P_d z_f^* = z_f^* \Rightarrow \xi(z^*(\cdot),P_d z_f^*) = \xi(z^*)$ and hence,

$$\mathcal{P}'\xi(z^*(\cdot),P_d z_f^*) = \mathcal{P}'\xi(z^*) = \xi(z^*)$$

since $\mathcal{P}'x = x$ when $x \in S$ . This implies that $z^* \in \nu(F) = f(\nu(F))$ .
Therefore,

$$z^* \in \nu(F) = f(\nu(F)) \Leftarrow z^* \in W_S^\varepsilon$$

that is, $F$ satisfies sufficient conditions. To see that $F$ also satisfies necessity we now suppose that $z^*$ is a fixed point of $F$, then

$$z^* = F(z^*) = \mathcal{S}z_0 + LNz^*(\cdot) + \mathcal{P}'\xi(z^*(\cdot),P_d z_f^*)$$

and hence

$$\mathcal{P}'\xi(z^*(\cdot),P_d z_f^*) = z^* - \mathcal{S}z_0 - LNz^*(\cdot) = \xi(z^*) \tag{6.4.17}$$

which implies that $\xi(z^*) \in S$ and therefore (6.4.1) is satisfied. Now observe that (6.4.17) is equivalent to

$$\mathcal{P}'(z^*(\cdot)-L(\cdot)Nz^*(\cdot)-\mathcal{S}(\cdot)z_0 \ , \ P_d z_f^*-L(T)Nz^*(\cdot)-\mathcal{S}(T)z_0) =$$

$$= (z^*(\cdot)-L(\cdot)Nz^*(\cdot)-\mathcal{S}(\cdot)z_0 \ , \ z_f^*-L(T)Nz^*(\cdot)-\mathcal{S}(T)z_0)$$

and, since $\mathcal{P}'$ is uniform in the first component, this implies that $P_d z_f^* = z_f^*$. Therefore (6.4.2) is also satisfied and $z^*$ is a wanted state. So,

$$z^* \in \nu(F) = f(\nu(F)) \Rightarrow z^* \in W_s^\varepsilon$$

i.e., $F$ has necessary condition too.

Summarizing, $F$ in (6.4.16) satisfies properties (a), (b), (c) and (e).

6.4.6 - Theorem.

Let $X$ be a normed vector space, $S'$ be a convex subset of $X$

with $0 \in S'$ , $P:X \to X$ be any projection onto $S'$ , $\eta:X \to X$ any mapping, $p:X \to \mathbb{R}$ a functional satisfying

$$0 < p(x) \leq 1 \tag{6.4.18}$$

and $F:X \to X$ given by

$$F(x) = x - \eta(x) + p(x)P\eta(x) . \tag{6.4.19}$$

Then, $\eta(x^*) \in S'$ and $p(x^*) = 1$ are sufficient conditions for $x^* \in \nu(F)$ , that is

$$x^* \in \nu(F) \quad \Leftarrow \begin{cases} \eta(x^*) \in S' \\ p(x^*) = 1 \end{cases} . \tag{6.4.20}$$

Moreover, if

$$\eta(x^*) \neq 0 \tag{6.4.21}$$

then $\eta(x^*) \in S'$ and $p(x^*) = 1$ are also necessary conditions for $x^* \in \nu(F)$ , that is

$$\left.\begin{array}{l} x^* \in \nu(F) \\ \eta(x^*) \neq 0 \end{array}\right\} \Rightarrow \begin{cases} \eta(x^*) \in S' \\ p(x^*) = 1 \end{cases} . \tag{6.4.22}$$

Furthermore, the result is also valid for $F$ given by

$$F(x) = x - \eta(x) + P(p(x)\eta(x)) . \tag{6.4.23}$$

Proof.

First consider the mapping $F$ in (6.4.19).

Necessity: Since $S'$ is convex and $0 \in S'$ and $p$ satisfies (6.4.18),

$$p(x)P\eta(x) \in S' \qquad \forall\, x \in X . \tag{6.4.24}$$

Now suppose $x^* \in \nu(F)$ and $\eta(x^*) \neq 0$ , thus

$$x^* = F(x^*) = x^* - \eta(x^*) + p(x^*)P\eta(x^*)$$

and hence

$$\eta(x^*) = p(x^*)P\eta(x^*) \tag{6.4.25}$$

which implies that $\eta(x^*) \in S'$ by (6.4.24). Therefore, $\eta(x^*) = P\eta(x^*)$ and (6.4.25) becomes

$$\eta(x^*) = p(x^*)\eta(x^*) \tag{6.4.26}$$

and, since $\eta(x^*) \neq 0$ , $p(x^*) = 1$ .

Sufficiency: Immediate since $P\eta(x^*) = \eta(x^*)$ when $\eta(x^*) \in S'$ . Now for $F$ in (6.4.23) the proof is analogous. In the necessity (6.4.25) is replaced by

$$\eta(x^*) = P(p(x^*)\eta(x^*)) . \tag{6.4.27}$$

Clearly (6.4.27) => $\eta(x^*) \in S'$ => $p(x^*)\eta(x^*) \in S'$ => $P(p(x^*)\eta(x^*)) =$ $= p(x^*)\eta(x^*)$ => (6.4.26).

In the sufficiency the result follows since

$$P(p(x^*)\eta(x^*)) = p(x^*)\eta(x^*)$$

for $p(x^*) = 1$ and $\eta(x^*) \in S'$ .

Q.E.D.

Remark.

In general $F$ in (6.4.19) and $F$ in (6.4.23) are two different mappings.

6.4.7 - New mappings with properties (b)-(e) satisfied.

Let $q:Z \to \mathbb{R}$ be given by either (2.1.4) or (6.4.6) with $m = 1$ . Thus

$$q(z_f) = 0 \iff z_f \in \overline{B_\varepsilon(z_d)}$$

and

$$0 \le q(z_f) < 1 .$$

Now set $p:X \to \mathbb{R}$ the functional

$$p(z) = 1 - q(z_f) , \quad z = (z(\cdot),z_f) .$$

Clearly $p$ satisfies

$$p(z) = 1 \iff z_f \in \overline{B_\varepsilon(z_d)} \tag{6.4.28}$$

and

$$0 < p(z) \le 1 .$$

Suppose that the control $u = 0$ is an admissible control, that is, $0 \in \mathcal{U}_{ad}$ (which is obviously the case when $\mathcal{U}_{ad} = \mathcal{U}$), where $\mathcal{U}_{ad}$ is assumed again to be either $\mathcal{U}_{ad} = \mathcal{U}$ or $\mathcal{U}_{ad}$ as in (6.1.2). Clearly we have that

$$0 \in S \quad \text{and} \quad S \text{ is convex}$$

and hence we can apply theorem 6.4.6 with

$$S' = S \ , \quad P = \mathcal{P} \quad \text{and} \quad \eta = \xi \ .$$

The mappings F in (6.4.19) and (6.4.23) become

$$F(z) = \mathcal{S}z_0 + LNz(\cdot) + p(z)\mathcal{P}\xi(z) \tag{6.4.29}$$

and

$$F(z) = \mathcal{S}z_0 + LNz(.) + \mathcal{P}(p(z)\xi(z)) \ . \tag{6.4.30}$$

Set $f = I$. From (6.4.20), since $p$ satisfies (6.4.28), we have

$$z^* \in f(\nu(F)) <= \begin{cases} \xi(z^*) \in S \\ z_f^* \in \overline{B_\varepsilon(z_d)} \end{cases}$$

which is the same as sufficient condition for F . From (6.4.22) we have that

$$\left.\begin{matrix} z^* \in f(\nu(F)) \\ \xi(z^*) \neq 0 \end{matrix}\right\} => \begin{cases} \xi(z^*) \in S \\ z_f^* \in \overline{B_\varepsilon(z_d)} \ . \end{cases} \tag{6.4.31}$$

Thus $F$ does not have necessary conditions. (6.4.31) could suggest us to consider the a posteriori test

$$\xi(z^*) \neq 0 \tag{6.4.32}$$

since each fixed point $z^* \in \nu(F) = f(\nu(F))$ which satisfies (6.4.32) will be a wanted state. However, we must not forget that fixed points $z^* = (z^*(\cdot), z_f^*)$ which do not satisfy (6.4.32) may also be a wanted state if $p(z^*) = 1$ (i.e., if $z_f^* \in \overline{B_\varepsilon(z_d)}$). In fact the set of wanted states is given by

$$W_s^\varepsilon = \{z^* \in \nu(F) : \xi(z^*) \neq 0\} \cup \{z^* \in \nu(F) : \xi(z^*) = 0 \text{ and } z_f^* \in \overline{B_\varepsilon(z_d)}\} .$$

Clearly we can obtain the wanted states once we have obtained $\nu(F)$. For the a posteriori test we can set (6.4.2), that is,

$$z^* \in \overline{B_\varepsilon(z_d)} \tag{6.4.33}$$

since every fixed point $z^*$ will satisfy (6.4.1), independent of the value of $\xi(z^*)$. Observe however that we consider $F$ (in either (6.4.29) or (6.4.30)) acceptable because the set of fixed points which fail the a posteriori test, given by

$$J_N(F) = \{z^* \in \nu(F) : \xi(z^*) = 0 \text{ and } z_f^* \notin \overline{B_\varepsilon(z_d)}\}$$

is much more restrictive than for the case of $F$ in (6.1.9).

Summarizing, both mappings $F$ given by (6.4.29) and (6.4.30) with a posteriori test (6.4.33) satisfy properties (b)-(e).

6.4.8 - Controllability to the origin.

We shall see here that in the particular case $z_d = 0$ we can obtain other classes of mappings $F$ which are simpler than the ones we have seen so far. Controllability to the origin often occurs in problems of stability.

First observe that when $z_d = 0$, the ball $\overline{B_\varepsilon(z_d)}$ becomes $\overline{B_\varepsilon(0)}$ and the projection $P_d$ can now be the $\varepsilon$-radial retraction (see example 4.1.5(i)). Set $p:X \to \mathbb{R}$ the functional

$$p(z) = 1 - \frac{\|\mathcal{P}\xi(z) - \xi(z)\|}{1 + \|\mathcal{P}\xi(z) - \xi(z)\|} .$$

Clearly $p$ satisfies (6.4.18), that is

$$0 < p(z) \leq 1$$

and, by property 4.1.2,

$$p(z) = 1 \iff \xi(z) \in S. \tag{6.4.34}$$

Now set

$$S' = \{z = (z(\cdot), z_f) \in X : z_f \in \overline{B_\varepsilon(0)}\} .$$

Clearly $0 \in S'$ and $S'$ is convex. One of the simplest projections $P:X \to X$ onto $S'$ is given by

$$P(z(\cdot),z_f) = (z(\cdot),P_d z_f) \quad .$$

We shall now apply theorem 6.4.6 with $\eta = I$ = identity on $X$ . The mapping $F$ in (6.4.19) becomes

$$F(z) = p(z)(z(\cdot),P_d z_f) \ . \tag{6.4.35}$$

By (6.4.20), since $p$ satisfies (6.4.34),

$$z^* \in \nu(F) <=\begin{cases} z_f \in \overline{B_\varepsilon(0)} \\ \xi(z^*) \in S \end{cases}$$

and hence $F$ has sufficient conditions if we set the operative function $f = I$ . Unfortunately $F$ does not have necessary conditions because there is one case, and that is

$$z^* = (z^*(\cdot),z_f^*) = 0 \quad ,$$

in which $z^* \in \nu(F)$ and $z^*$ may not be a wanted state. In all other cases we have

$$z^* \in \nu(F) => z^* \in W_S^\varepsilon \ .$$

Now observe that for $0 \in X$ , (assuming that the nonlinearity $N$ satisfies $N(0) = 0$) ,

$$0 \in W_S^\varepsilon <=> z_0 = 0$$

and therefore the set of wanted states is given by

$$W_s^{\varepsilon} = \{z^* \in \nu(F) : z^* \neq 0\} \cup \Omega$$

where $\Omega$ is the set

$$\Omega = \begin{cases} \emptyset & \text{if } z_0 \neq 0 \\ \{0\} & \text{if } z_0 = 0 . \end{cases}$$

So the a posteriori check here is to check if each fixed point is different from zero but do not eliminate $z^* = 0$ if the initial state $z_0 = 0$ . Summarizing F in (6.4.35) satisfies properties (b)-(e) and in the particular case $z_0 = 0$ , F satisfies all properties (a)-(e).

6.4.9 - Property (b) dropped.

If we drop property (b), that is, if we do not require F to have sufficient conditions, this only means that we cannot obtain all the wanted states from the fixed points of F . Clearly if only one wanted control $u^* \in \mathcal{U}_{ad}$ is required then it is enough to obtain one wanted state and in this case property (b) is no longer important.

Let $S_1$ be any closed and convex subset of S and $P_1 : X \to X$

$$P_1 = \text{any continuous projection onto } S_1 . \qquad (6.4.36)$$

Apart from property (b), the results obtained for every mapping F in this section continue valid if we replace $P$ by $P_1$ . In fact, when $P$ is replaced by $P_1$ we reduce the set of wanted states which may be obtained from $f(\nu(F))$ to the subset of $W_s^{\varepsilon}$

$$\{z^* = (z^*(\cdot), z^*_f) \in X : \xi(z^*) \in S_1 \quad \text{and} \quad z^*_f \in \overline{B_\varepsilon(z_d)}\}$$

and the set $J_S(F)$ defined in (6.1.10) becomes

$$J_S(F) = \{z^* = (z^*(\cdot), z^*_f) : \xi(z^*) \in S \backslash S_1 \text{ and } z^*_f \in \overline{B_\varepsilon(z_d)}\} .$$

We can use the present approach in cases such as for example a projection $P$ onto $S$ is not immediate. Then, with a suitable choice of the subset $S_1 \subset S$ (e.g. $S_1$ = some ball in $X$), a more immediate projection $P_1$ onto $S_1$ may be readily available.

Another possible relaxation when we drop property (b) is to consider the mapping $F: D \to X$ for some $D \subset X$, rather than $F$ defined in the whole $X$. In this case we are only considering the wanted states $z^* \in W^\varepsilon_s$ which satisfy $z^* \in D$ and $J_S(F)$ is the set

$$J_S(F) = \{z^* \in W^\varepsilon_s : z^* \notin D\} .$$

Actually this approach has to be used when applying fixed point theorems which are stated for mappings $F: D \to X$ with $D$ bounded.

6.4.10 - <u>More general set of admissible controls $\mathcal{U}_{ad}$</u>.

The set of admissible controls $\mathcal{U}_{ad}$ may not satisfy either the conditions (6.1.2) or (6.1.3). For example, the simple case of $U = \mathbb{R}$, $\mathcal{U} = L^2(0,T) = L^2(0,T;U)$ and

$$\mathcal{U}_{ad} = \{u(\cdot) \in \mathcal{U} : u(t) \geq 0 \quad \text{a.e.}\} .$$

If we can find a subset $\mathcal{U}_1 \subset \mathcal{U}_{ad}$ such that

$\mathcal{U}_1$ is bounded, closed and convex

then, setting

$$S_1 = LB(\mathcal{U}_1)$$

and $P_1$ as in (6.4.36) we can extend the result obtained in the present section to this case with the sacrifice of property (b). (See §6.4.9.)

## 6.5 - THE SEARCH FOR THE SOLUTIONS.

### 6.5.1 - Fixed point theorems.

When we want to determine the existence of a solution for our problem, i.e., the existence of wanted states, then the fixed point theorems (see section 2.6) play an important role by giving conditions in which F has fixed points. Obviously the choice of the fixed point theorem to apply will depend on the type of nonlinearity N . Also note that in most fixed point theorems it is necessary to consider $F:D \to X$ , with D being a bounded set where the conditions for the particular fixed point theorem used have to be satisfied. That is, depending on the fixed point theorem employed, D has to be chosen accordingly.

Some of the mappings F in section 6.4 had the form

$$F(z) = Sz_0 + LNz(\cdot) + P\eta(z) \tag{6.5.1}$$

where $\eta(z)$ is $\xi(z)$, $\xi(z(\cdot),z_d)$, $\xi(z(\cdot),P_d z_f)$ , etc... Suppose that the nonlinearity N satisfies (2.2.6) for some $k_N$ and $\overline{k}_N$ and L satisfies (2.2.13) and (2.2.14) for some $k_1'$ and $k_2'$ . This implies that LN satisfies (2.2.15)-(2.2.18) for the constants $c_1,c_2,\overline{c}_1$ and $\overline{c}_2$ given by (see §2.2.3)

$$c_1 = k_N k_1' \ , \ c_2 = k_N k_2' \ , \ \overline{c}_1 = \overline{k}_N k_1' \quad \text{and} \quad \overline{c}_2 = \overline{k}_N k_2' \ .$$

We have seen in section 2.6 that most fixed point theorems require that F maps D into D . This could be achieved in several ways by a careful choice of D , for example, let $\mathcal{U}_{ad}$ be bounded, as in the case (6.1.2), then S is bounded. Suppose that $z_0 = 0$ and

$$S \subset B_a(0) \quad \text{in} \quad X = \mathcal{M}^s(0,T;Z)$$

then

$$\|\mathcal{P}\eta(z)\|_{\mathcal{M}^s} \le a \ . \tag{6.5.2}$$

In this case we have that, if for some $r > a$

$$\overline{c}_1 \le \frac{r-a}{r} \quad \text{and} \quad \overline{c}_2 \le \frac{r-a}{r}$$

and $D = B_r(0)$ , then F maps D into D .

To see this first note that for $z \in X$ , if $\mathcal{P}\eta(z) = (\tilde{x}(\cdot),\tilde{x}_f)$ , then (6.5.2) implies that

$$\|\tilde{x}(\cdot)\|_{H^s} \le a \quad \text{and} \quad \|\tilde{x}_f\|_Z \le a$$

and hence

$$\|F(z)\|^2_{M^S} = \|L(\cdot)Nz(\cdot)-\tilde{x}(\cdot)\|^2_{H^S} + \|L(T)Nz(\cdot)-\tilde{x}_f\|^2_Z$$

$$\leq (\bar{c}_1 r + a)^2 + (\bar{c}_2 r + a)^2$$

$$\leq r^2$$

and therefore $F(z) \in B_r(0)$ in $X = M^S(0,T;Z)$ .

If $z_0 \neq 0$ similar results could be obtained using $D = B_r(Sz_0)$ . In the case of the mapping $F$ given by (6.4.35), namely

$$F(z) = p(z)\ (z(\cdot),P_d z_f)$$

we have that if $r' > 0$ and $r'' > \varepsilon$ and

$$D = B_{r'}(0) \times B_{r''}(0) \subset M^S(0,T;Z)$$

where the ball $B_{r'}(0)$ is in $H^S(0,T;Z)$ and the ball $B_{r''}(0)$ is in $Z$ , then $F$ maps $D$ into $D$ .

Now observe that $F$ in (6.5.1) can be expressed in the form $F = F_1+F_2$ where

$$F_1(z) = Sz_0 + LNz(\cdot)$$

$$F_2(z) = Pn(z)$$

which is particularly convenient for fixed points of contractive type with perturbation (see section 2.6) if

$F_1$ is a contractive type mapping (6.5.3)

$F_2$ is completely continuous (6.5.4)

$F_1$ satisfies (6.5.3) if, for example $k = \sqrt{c_1+c_2} \le 1$. By theorem 4.1.4, $F_2$ will satisfy (6.5.4) if $S$ is compact, which is the case if $\mathcal{U}_{ad}$ is compact. If this is not the case however, we can take $S_1 \subset S$, $S_1$ compact and $P = P_1$ = continuous projection onto $S_1$ (see §6.4.9) and then $F_2$ will satisfy (6.5.4).

It is also possible to use the approach of projections in other standard procedures in numerical analysis, (such as for example, minimization of functionals or solving the operator equation of the type $G(x) = 0$, etc.) in order to find the set of wanted states $W_s^\varepsilon$.

6.5.2 - Optimization of functionals.

A well known result in functional analysis [10,23] is that if $D$ is a compact set in $X$ and $g: X \to \mathbb{R}$ is lower semi-continuous on $D$, then $g$ achieves a minimum in $D$. We could use this result, for example, to seek the wanted states in $D$. There are well known methods, such as the steepest descent [23], used in minimum norm problems.

Let $\overline{p}: X \to \mathbb{R}$ be a continuous functional which satisfies

$$\overline{p}(z) \ge 0 \qquad \forall\, z \in X \tag{6.6.1}$$

$$\overline{p}(z) = 0 \iff \xi(z) \in S . \tag{6.6.2}$$

Using any continuous projection $P$ onto $S$ we have that the functional $\overline{p}$ given by

$$\bar{p}(z) = \|\mathcal{P}\xi(z) - \xi(z)\|_X$$

satisfies (6.6.1)-(6.6.2). Other $\bar{p}$ could be obtained similarly to $p$ in §6.4.8. Now let $q:Z \to \mathbb{R}$ be a continuous functional which satisfies

$$q(z_f) \geq 0 \qquad \forall\, z_f \in Z \tag{6.6.3}$$

$$q(z_f) = 0 \iff z_f \in \overline{B_\varepsilon(z_d)} \ . \tag{6.6.4}$$

We have already seen that for $P_d$ any continuous projection onto $\overline{B_\varepsilon}(z_d)$ , the functional $q$ given by

$$q(z_f) = \|P_d z_f - z_f\|_Z$$

satisfies (6.6.3)-(6.6.4) and so does $q$ in (6.4.6) and also $q$ in (2.1.4).

Setting $g:X \to \mathbb{R}$

$$g(z) = \bar{p}(z) + q(z_f) \quad , \quad \forall\, z = (z(\cdot), z_f) \in X$$

we have that $g$ is continuous (and hence lower semi-continuous too) and satisfies

$$g(z) \geq 0$$

$$g(z) = 0 \iff z \in W_s^\varepsilon \ .$$

Clearly if $z^*$ is a wanted state then $g$ achieves a minimum at $z^*$ . On the other hand, if $g$ achieves a local minimum $z^*$ then, provided

$z^*$ is also a global minimal (i.e., $g(z^*) = 0$), then $z^*$ is a wanted state.

### 6.5.3 - The operator equation $G(x) = 0$ .

There are some special iterative schemes [10,19,23] developed to find the solution of the operator equation

$$G(x) = 0 \tag{6.6.5}$$

where $G \in \mathcal{O}p(X_1,X_2)$ for two metric spaces $X_1$ and $X_2$ . Among these schemes are the well known (classical) Newton's method [10,19,23] and the modified Newton-Kantorovic method [10,19].

Set $X_1 = X_2 = X$ and let $G:X \to X$ be given by

$$G(x) = F(x) - x$$

for some mapping $F:X \to X$ of section 6.4. Clearly now we have that any solution $x^*$ of (6.6.5) is a fixed point of $F$ .

## 6.6 - MORE GENERAL TYPES OF SYSTEMS.

The results of this chapter were developed for systems of the type

$$\dot{z}(t) = Az(t) + Bu + Nz(t) \;, \quad z(0) = z_0$$

where $A$ generates a strongly continuous semigroup $S(\cdot)$ . This enabled us to see how the methods presented progressively improved since the first attempts to use fixed points in controllability. However the same approach of projections could be applied to more general nonlinear systems. Consider for example the following system (in its mild form):

$$z(t) = f_0(z_0,t) + \int_0^t f_1(z(\cdot),s,t)ds + \int_0^t f_2(z(\cdot),s,t,Bu(\cdot))ds \qquad (6.6.1)$$

$$z(0) = z_0$$

where the functions $f_0, f_1$ and $f_2$ have their values in a space $\mathfrak{F}$ of functions from $[0,T]$ to $Z$ (e.g. $\mathfrak{F} = L^2(0,T;Z)$) and $Z$ is the state space.

If we define

$$z_f = z(T) = f_0(z_0,T) + \int_0^T f_1(z(\cdot),s,T)ds + \int_0^T f_2(z(\cdot),s,T,Bu(\cdot))ds \qquad (6.6.2)$$

then (6.6.1)-(6.6.2) give us the pair $(z(\cdot),z_f)$, trajectory-final state which lies in $X = \mathfrak{F} \times Z$. Clearly the results of this chapter have a straightforward generalization for systems of the above type. The operator $LB:U \to X$ is replaced by the operator $T:U \to X$

$$Tu = \left(\int_0^\cdot f_2(z(\cdot),s,\cdot,Bu(\cdot))ds, \int_0^T f_2(z(\cdot),s,T,Bu(\cdot))ds\right)$$

and the set $S$ becomes

$$S = T(U_{ad}) .$$

Moreover, the constant term $Sz_0$ which appeared so often would be substituted by $x_0 \in X$ given by

$$x_0 = (f_0(z_0,\cdot),f_0(z_0,T)) .$$

## 6.7 - SOME EXAMPLES OF THE ADJUSTMENT OF THE SPACES $\mathcal{U}$ AND $\mathcal{X}$ .

In this chapter we have assumed (see §6.1.1) that if $\mathcal{U}_{ad} = \mathcal{U}$ (or $\mathcal{U}_{ad}$ is a closed subspace of $\mathcal{U}$) , then (5.10.4) holds, that is

$$\text{LB has closed range,} \tag{6.7.1}$$

whereas if $\mathcal{U}_{ad}$ is a bounded, closed and convex set, then (5.10.5) holds, that is,

$$\text{LB is bounded and has closed range.} \tag{6.7.2}$$

We have seen in section 5.10 that the above conditions guarantee the existence of continuous projections $\mathcal{P}$ onto $S = LB(\mathcal{U}_{ad})$ . We have also seen in §§5.10.3 and 5.10.4 two different procedures to adjust the spaces $\mathcal{U} = H^r(0,T;U)$ for some $r \in \mathbb{R}$ (or $H_0^r(0,T;U)$ for some $r \geq 0$) and/or $\mathcal{X} = \mathcal{M}^s(0,T;Z)$ for some $s \in \mathbb{R}$ . Here we present some specific examples.

As observed in section 5.10, in general a large space of input functions (such as $\mathcal{U} = L^2(0,T;U)$ or $\mathcal{U} = H^{-1}(0,T;U)$) is not desirable since it may contain discontinuous functions or even distributions, which are unsuitable for practical applications. We have also noted in section 5.10 that we can have condition (6.7.1) or (6.7.2) satisfied for a smooth space $\mathcal{U}$ (such as $\mathcal{U} = H^1(0,T;U)$ or $\mathcal{U} = H_0^1(0,T;U)$) by restricting the space $\mathcal{X}$ to pairs of trajectory-final state which are even smoother (such as $\mathcal{X} = \mathcal{M}^s(0,T;Z) = H^s(0,T;Z) \times Z$ for some $s > 1$) . In fact the results of chapter 5 are a powerful tool in the construction of these spaces $\mathcal{U}$ and $\mathcal{X}$ .

6.7.1 - First example:

Consider the following nonlinear system on a Hilbert space $Z$

$$\frac{dz}{dt}(t) = f(\frac{dz}{dt}, z(t), t) + u \qquad z(0) = z_0 \tag{6.7.3}$$

where the controls $u \in \mathcal{F}(0,T;Z)$ (i.e. $u$ is a function from $[0,T]$ to $Z$, see page 8 for definition of $\mathcal{F}(D,Z)$). Set $N$ the nonlinear operator $z(\cdot) \to Nz(\cdot)$

$$Nz(t) = f(\frac{dz}{dt}, z(t), t)$$

then we can write system (6.7.1) in the form (3.1.1), i.e.,

$$\frac{dz}{dt} = Nz(\cdot) + \mathcal{B}u \qquad z(0) = z_0 \tag{6.7.4}$$

where $\mathcal{B} = I_U$ (identity on $U = Z$) and $A = 0$ (i.e., no linear part or linear part incorporated to the nonlinear part). This is a relatively simple example of nonlinear controllability, nevertheless it will be interesting to see how our approach can be used to solve it.

Since $\mathcal{B} = I_U$, $B = I$ (identity on $\mathcal{U}$) (see section 3.1). Since $A = 0$, $S(t)z_0 = z_0$ for all $t$ and for $L(\cdot)$ defined by (2.2.11), $L(\cdot)B$ becomes the operator

$$u \to \int_0^{\cdot} u(s)ds \ .$$

The mild form of (6.7.4) is given by (3.2.5), namely

$$z(\cdot) = z_0 + L(\cdot)Nz(\cdot) + L(\cdot)Bu \qquad\qquad z(0) = z_0$$

where here we identify $z_0$ with the function $t \to z_0$, from $[0,T]$ to $Z$. Now set $X = M^S(0,T;Z)$,

$$z_f = z(T) \qquad \text{and} \qquad z = (z(\cdot),z_f) \in X \ .$$

Then we can write the system in the form (6.1.1) as

$$z = (z_0,z_0) + LNz(\cdot) + LBu \ , \qquad z(0) = z_0 \ . \tag{6.7.5}$$

Let us now concentrate on the operator $LB:U \to X$,

$$LBu = \left(\int_0^{\cdot} u(s)ds, \int_0^T u(s)ds\right) \ . \tag{6.7.6}$$

First we want to have condition (6.7.2) satisfied, i.e., $LB$ is bounded and has closed range. We shall use the second procedure (developed in §5.10.4) to achieve this. Take the following linearly independent set $\{e_{nm}\}_{(n,m)\in\Delta}$ of functions from $[0,T]$ to $Z$, $\Delta = \mathbb{N} \times \mathbb{Z}$:

$$e_{nm}(t) = \begin{cases} \cos(2m\pi/T)t.\psi_n & n \in \mathbb{N} \ , \ m \in \mathbb{Z}^+ = \{0,1,2,\dots\} \\ \sin(2|m|\pi/T)t.\psi_n & n \in \mathbb{N} \ , \ m \in \{-1,-2,\dots\} \end{cases}$$

where $\{\psi_n\}_{n\in\mathbb{N}}$ is any c.o.s. in $Z$. Then $LBe_{nm}, (n,m) \in \Delta$ is given by

$$LBe_{nm} = \begin{cases} ( \quad t \quad , \ T).\psi_n & \text{for } n \in \mathbb{N}, \ m = 0 \\ (\frac{T}{2m\pi} .\sin(\frac{2m\pi}{T})t \ , \ 0).\psi_n & \text{for } n \in \mathbb{N}, \ m = 1,2,\ldots \\ (\frac{T}{2m\pi} .\cos(\frac{2m\pi}{T})t \ , \ 0).\psi_n & \text{for } n \in \mathbb{N}, \ m = -1,-2,\ldots \end{cases}$$

and therefore,

$$R = \{LBe_{nm}\}_{(n,m)\in\Delta}$$

is linearly independent. Set $U = U'(e_{nm},\alpha_{nm},\Delta)$, that is, the input functions $u \in U$ are given by

$$u = \sum_{n\in\mathbb{N}} \Big( \sum_{m\in\mathbb{Z}^+} u_{nm} \cos(\frac{2m\pi}{T})t + \sum_{m\in\mathbb{N}} u_{nm} \sin(\frac{2m\pi}{T})t\Big)\psi_n$$

with norm

$$\| u \|^2_U = \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{Z}} |u_{nm}|^2 \alpha_{nm} .$$

By theorem 5.9.3, condition (6.7.2) is satisfied if we set

$$\alpha_{nm} = \| LBe_{nm} \|^2_{M^s(0,T;Z)} \tag{6.7.7}$$

For example, for $s = 0$, $X = M^0(0,T;Z) = L^2(0,T:Z) \times Z$ and $\alpha_{nm}$ becomes

$$\alpha_{n0} = \frac{T^2(T+3)}{3}\ , \ n \in \mathbb{N} \quad \text{and} \quad \alpha_{nm} = \frac{T^2}{4m^2\pi^2} \qquad n \in \mathbb{N}\ , \quad m = \pm 1, \pm 2, \ldots$$

and by the imbedding corollary 5.7.8 it is easy to verify that

$$\mathcal{U} \simeq H^{-1}(0,T;Z)\ .$$

So $LB$ is bounded and has closed range from $H^{-1}(0,T;Z)$ to $M^0(0,T;Z)$. Now suppose we set $X$ a smoother space. For example, $s = 1$ and hence $X = M^1(0,T;Z) = H^1(0,T;Z) \times Z$. With $\alpha_{nm}$ defined by (6.7.7) $\mathcal{U}$ becomes a smoother space too. Actually, by the imbedding corollary 5.7.8,

$$\mathcal{U} \simeq L^2(0,T;Z)\ .$$

So $LB{:}L^2(0,T;Z) \to M^1(0,T;Z)$ is bounded and has closed range.

Now set $X$ even smoother. For example $s = 2$ and thus $X = M^2(0,T;Z) = H^2(0,T;Z) \times Z$. With $\alpha_{nm}$ defined by (6.7.7) again $\mathcal{U}$ becomes even smoother too. In fact we now have

$$\mathcal{U} \simeq H^1(0,T;Z)\ .$$

So $LB{:}H^1(0,T;Z) \to M^2(0,T;Z)$ is bounded and has closed range. It is easy to obtain the following generalization of this result:

$$LB{:}H^{s-1}(0,T;Z) \to M^s(0,T;Z) \quad \text{is bounded and has closed range.}$$

This illustrates what we have already mentioned that we can obtain condition (6.7.2) satisfied for smooth spaces $\mathcal{U}$ by restricting $X$ to smoother spaces. This will be a common pattern in every example shown here.

It is also easy to deduce that $LB:\mathcal{U} \to X$ satisfies condition (6.7.1) for $\mathcal{U} = H^r(0,T;Z)$ and $X = \mathcal{M}^s(0,T;Z)$ if $r \le s-1$ . This follows since, by theorem 5.9.3 again, if

$$\alpha_{nm} \le \| LBe_{nm} \|_X^2 \qquad \text{for all} \quad (n,m) \in \Delta \quad ,$$

then $LB:\mathcal{U} \to X$ has closed range.

Note that in the present example we obtained spaces $\mathcal{U}$ and $X$ for which conditions (6.7.1) or (6.7.2) hold by following, step by step, the second procedure (presented in §5.10.4). We could also have obtained the same results by following the first procedure (presented in §5.10.3). Also observe that, we did not need to construct the representation of the space $X$ , as shown in §5.9.2. However, such representation could be useful and therefore we present here how it would be for the present example:

First, to ease the notation, let $\eta: \mathbb{N} \to \mathbb{N} \times \{\mathbb{Z} \setminus \{0\}\}$ be any bijective map which sets

$$j \to (n_j, m_j) \in \mathbb{N} \times \{\mathbb{Z} \setminus \{0\}\} \qquad \text{for each} \quad j \in \mathbb{N} .$$

Now set $\Phi_{0j}(\cdot)$ the sequence (of sinces and cosines) given by

$$\Phi_{0j}(t) = \begin{cases} \dfrac{T}{2m_j\pi} \cdot \sin(\dfrac{2m_j\pi}{T})t \cdot \psi_{nj} & \text{if} \quad m_j = 1,2,\ldots \\[2ex] \dfrac{T}{2m_j\pi} \cdot \cos(\dfrac{2m_j\pi}{T})t \cdot \psi_{nj} & \text{if} \quad m_j = -1,-2,\ldots \end{cases}$$

and $\Phi_{1j}(\cdot)$ given by

$$\Phi_{1j}(t) = t\,\psi_j \qquad \text{for all } j \in \mathbb{N} .$$

Now observe that, since $\{\Phi_{0j}(\cdot)\}_{j\in\mathbb{N}} \cup \{\Phi_{1j}(\cdot)\}_{j\in\mathbb{N}}$ is complete in $H^s(0,T;Z)$ and $\{T\psi_n\}_{n\ \mathbb{N}}$ is complete in $Z$, we have that

$$E_2 = E_4 = \emptyset$$

and

$$X = M^s(0,T;Z) \simeq E_0 \oplus E_1 \oplus E_3$$

where

$$E_i = X'(\Phi_{ij}\ ,\ \beta_{ij}\ ,\ \mathbb{N}) \qquad i = 0,1,3$$

$$\Phi_{0j} = (\Phi_{0j}(\cdot)\ ,\ 0\ ) \qquad j \in \mathbb{N}$$

$$\Phi_{1j} = (\Phi_{1j}(\cdot)\ ,\ T\psi_j\ ) \qquad j \in \mathbb{N}$$

$$\Phi_{3j} = (\Phi_{1j}(\cdot)\ ,\ 0\ ) \qquad j \in \mathbb{N}$$

and

$$\beta_{0j} = \|\Phi_{0j}\|^2_{M^s} = \frac{T^2}{4m_j^2\pi^2}\,\|\sqrt{2}\,\sin(\frac{2m_j\pi}{T})t\,\|^2_{H^s(0,T)} \qquad j \in \mathbb{N}$$

$$\beta_{1j} = \|\Phi_{1j}\|^2_{M^s} = \|t\|^2_{H^s(0,T)} + T^2 \qquad j \in \mathbb{N}$$

$$\beta_{3j} = \|\Phi_{3j}\|^2_{M^s} = \|t\|^2_{H^s(0,T)} \qquad j \in \mathbb{N} .$$

In other words, if $z \in X = M^S(0,T;Z)$ then $z$ can be expressed as

$$z = \sum_{i=0,1,3} \sum_{j \in \mathbb{N}} a_{ij} \Phi_{ij}$$

and $\|\cdot\|_{M^S}$ is equivalent to the following norm

$$\|z\|^2 = \sum_{i=0,1,3} \sum_{j \in \mathbb{N}} |a_{ij}|^2 \alpha_{ij} \quad .$$

This representation of $M^S(0,T;Z)$ gives simple expressions for $LB$, $L(\cdot)B$ and $L(T)B$ as well as for $(LB)^\dagger$, the projections $\Pi$, $\Pi_1$, $\Pi_2$, etc. For example

$$\Pi z = \sum_{i=0,1} \sum_{j \in \mathbb{N}} a_{ij} \Phi_{ij} \; .$$

Now let us see a few remarks regarding this example before we move to further examples.

Remark A:

Note that there would be several other options for $\Delta$ and $\{e_{nm}\}_{(n,m)\in\Delta}$ different from the one chosen here. For example,

$$e_{nm}(t) = \sin(m\pi/T)t\,\psi_n \qquad (n,m) \in \Delta = \mathbb{N} \times \mathbb{N}$$

or

$$e_{nm}(t) = t^m \psi_n \qquad (n,m) \in \Delta = \mathbb{N} \times \mathbb{Z}^+$$

etc. Different choices of $e_{nm}$ will give different representation of the space of input functions $\mathcal{U}$. In any case, we must have

$$LBe_{nm} \in \mathcal{M}^s(0,T;Z) \qquad (n,m) \in \Delta \quad . \tag{6.7.8}$$

That is, when choosing $e_{nm}$ we have to be careful since $LBe_{nm}$ must belong to $X$ for $X = \mathcal{M}^s$ chosen for each $(n,m) \in \Delta$, so that we can calculate $\alpha_{nm} = \| LBe_{nm} \|^2_{\mathcal{M}^s}$. This, in general, will not bring any problem in the cases where $e_{nm}$ is a sequence of sines, cosines or monomials such as the above ones. However, an arbitrary choice of $e_{nm}$ could bring problems. For example, let $\{e_{nm}\}_{(n,m)\in\Delta}$ be the sequence chosen for this example except for $e_{n0}$, which we set

$$e_{n0}(t) = f(t)\psi_n \qquad n \in \mathbb{N} .$$

Both $\{e_{nm}\}_{(n,m)\in\Delta}$ and $R = \{LBe_{nm}\}_{(n,m)\in\Delta}$ are linearly independent if and only if $f:[0,T] \to Z$ satisfies

$$\int_0^T f(t)dt \neq 0 . \tag{6.7.9}$$

If we choose $f$ such that $f \in H^s(0,T)$ but $f \notin H^{s+1}(0,T)$ then we shall not be able to have $LBe_{nm} \in \mathcal{M}^s(0,T;Z)$. If the choice of $f$ however is such that (6.7.9) is satisfied and $f \in H^{s+1}(0,T)$, then $LBe_{nm} \in \mathcal{M}^s(0,T;Z)$ for all $(n,m) \in \Delta$.

□

Remark B:

If we define

$$\tilde{e}_{0j} = e_{n_j,m_j} \qquad \text{for all} \quad j \in \mathbb{N}$$

$$\tilde{e}_{1j} = e_{n_j,0} \qquad \text{for all} \quad j \in \mathbb{N}$$

$\Gamma = \{0,1\} \times \mathbb{N}$ and $\Lambda = \{0,1,2,3,4\} \times \mathbb{N}$

then

$$M = (\{\tilde{e}_{ij}\}_{(i,j)\in\Gamma} , \{\Phi_{ij}\}_{(i,j)\in\Lambda} ,\Gamma,\Gamma,\Lambda)$$

is a complete matched set for the primitive operator $(LB)^1:[U] \to X = M^S(0,T;Z)$ of LB . So, we have actually constructed a complete matched set for $(LB)^1$ . The fact that R is linearly independent allowed us to obtain $M$ without having to use any of the methods presented in section 5.8 (The generation of complete matched sets).

We shall see in the next examples that $R = \{LBe_k\}_{k\in\Delta}$ will often be linearly independent for some linearly independent sequence $\{e_k\}_{k\in\Delta}$ of functions $e_k:[0,T] \to U$ . However, in general we can always generate a complete matched set, say

$$M = (\{e_k\}_{k\in\Gamma} , \{\Phi_k\}_{k\in\Lambda} ,\Delta,\Gamma,\Lambda)$$

for the primitive operator of LB by applying the techniques presented in section 5.8. Once $M$ is obtained, both $\{e_k\}_{k\in\Delta}$ and

$$R = \{\Phi_k\}_{k\in\Delta} = \{LBe_k\}_{k\in\Delta}$$

are linearly independent sets and (6.7.8) holds. □

Remark C:

In this example we have that by setting $\mathcal{X} = \mathcal{M}^s$ , $s = r+1$ and $\alpha_{nm}$ as in (6.7.7) then

$$\mathcal{U} = H^r(0,T;U) \qquad \text{for} \quad U = Z ,$$

that is, we can make $\mathcal{U}$ as smooth as we like (with respect to t) and as smooth as Z (with respect to U).

In the next examples we shall also be able to make $\mathcal{U}$ as smooth as we like (in t) . However, here we could make $\mathcal{U}$ as smooth as Z (with respect to U) because

$$L(T)B : L^2(0,T;Z) \to Z$$

has closed range in Z . We shall see that if, for example, L(T)B , regarded as an operator from $L^2(0,T;Z)$ to Z , has range dense in Z but $\neq Z$ , i.e.,

$$\mathcal{R}(L(T)B) = Z' \subset Z , \qquad Z' \text{ smoother than } Z ,$$

then the space of input functions $\mathcal{U}$ , constructed similarly to here, will be smoother (in t) than $L^2(0,T;Z)$ but larger (in U) than Z . That is $\mathcal{U}$ = some $H^r(0,T;\bar{Z})$ , $r > 0$ but $\bar{Z} \supset Z$ . Actually we should expect this since by setting $\alpha_{nm}$ as in (6.7.7), not only

$$LB:\mathcal{U} \to \mathcal{X} = \mathcal{M}^s(0,T;Z) \quad \text{has closed range}$$

but also

$$L(T)B:\mathcal{U} \to Z \quad \text{has closed range}$$

(see §5.10.5). Clearly if

$$L(T)B:L^2(0,T;Z) \to Z \text{ does not have closed range}$$

$\mathcal{U}$ will have to be large (with respect to U) than Z .

In any case, if we take a new state space $Z' \subset Z$ such that for some U (e.g. U = Z) ,

$$L(T)B:L^2(0,T;U) \to Z' \text{ has closed range} \tag{6.7.10}$$

and redefine $\alpha_{nm}$ as

$$\alpha_{nm} = \| LBe_{nm} \|^2_{M^s(0,T;Z')} = \| L(\cdot)Be_{nm} \|^2_{H^s(0,T;Z')} + \| L(T)Be_{nm} \|^2_{Z'}$$

then we shall be able to obtain spaces of input functions $\mathcal{U}$ which are as smooth as we like (in t) and not larger than Z (with respect to U) .

So, the adjustment will not be just of the spaces $\mathcal{U}$ and $X$ but in fact of the spaces

$$\mathcal{U}\ , \quad Z' \quad \text{and} \quad X = M^s(0,T;Z') \ .$$

Moreover, the nonlinearity N and the initial state $z_0$ will also have to be considered in the adjustment. Additional conditions may have to be imposed in either N or $z_0$ in order to take $X = M^s$ smoother and smoother. (See Remarks E and F.) □

Remark D:

Observe that, although here $B:U = Z \to Z$ is bounded (since $B = I_U = I_Z$) , the operator

$$B:\mathcal{U} = H^r(0,T;Z) \to X = M^s(0,T;Z)$$

is not bounded if $r > s$ . Nevertheless, if $\alpha_{nm}$ is given by (6.7.7) then

$$LB:\mathcal{U} \to X = M^s(0,T;Z) \text{ is continuous (i.e. bounded)}$$

(and has closed range too). We have already seen in example 2.2.5 that if $B \in \hat{\mathcal{L}}(\mathcal{U},\mathcal{Y})$ for some space $\mathcal{Y}$, possibly larger than $X$ , and $LB = (L(\cdot)B,L(T)B) \in \mathcal{L}(\mathcal{Y},X)$ , then $LB \in \mathcal{L}(\mathcal{U},X)$ , that is $LB:\mathcal{U} \to X$ is continuous.

Remark E:

Similarly, we can have

$$LN:H^s(0,T;Z) \to X = M^s(0,T;Z) \text{ is continuous}$$

for nonlinearities $N$ which are continuous from $H^s(0,T;Z)$ to some larger space $\mathcal{Y}$ , as long as $L(\cdot)$ satisfies (2.2.13) and $L(T)$ satisfies (2.2.14). In the present example the above condition holds for any continuous linear operator $N$ from $H^s(0,T;Z)$ to $\mathcal{Y} = H^{s-1}(0,T;Z)$ .

In the next examples, once we have selected $\mathcal{U}$, $Z'$ and $X = M^s(0,T;Z')$ (see Remark C), the nonlinear part must satisfy

$$L(\cdot)N:H^s(0,T;Z') \to H^s(0,T;Z') \text{ is continuous} \tag{6.7.11}$$

then

$$LN:H^s(0,T;Z') \to M^s(0,T;Z') \text{ is continuous}$$

and system (6.1.1) is well defined in the adjusted space $X = M^s(0,T;Z')$ .

Observe that (6.7.11) is a very loose condition. We only require continuity and $N(0)$ is possibly $\neq 0$. In the sequel, in some cases we shall not impose any additional conditions to the nonlinearity.

Remark F:

Finally note that, in order to have system (6.1.1) well defined in the adjusted space $X = M^s(0,T;Z')$ the initial state $z_0$ must satisfy

$$S(\cdot)z_0 \in H^s(0,T;Z')$$

so that $Sz_0 \in X$. Clearly this is the case in the present example.

Condition (6.7.10) is satisfied in several cases such as for example, if $z_0 = 0$; or if the state is finite dimensional; or also if $A \in \mathcal{L}(Z')$, since in this case we have that (see §2.3.1)

$$\frac{d^n S}{dt^n}(t)z_0 = A^n S(t)z_0 \qquad \text{for all } z_0 \in Z' \text{ , } t > 0 \text{ , } n = 1,2,\ldots$$

and therefore $S(\cdot)z_0 \in H^s(0,T;Z')$ for any $s$.

If $A$ is unbounded however, then (6.7.12) holds for $s \leq 0$. Also, for $s = 1,2,\ldots$, (6.7.12) holds if

$$z_0 \in \mathcal{D}(A^s) . \qquad \square$$

6.7.2 - Second example:

The example of the previous section has a straightforward generalization for system described by

$$\frac{d^k z}{dt^k} = f\left(\frac{d^k z}{dt^k}, \ldots, \frac{dz}{dt}, z(t), t\right) + u(t)$$
$$z(0) = z_0 , \frac{dz}{dt}(0) = z_0^1 , \ldots, \frac{d^k z}{dt^k}(0) = z_0^k , \tag{6.7.12}$$

on a Hilbert space $Z$ again. Setting $z(\cdot) \mapsto Nz(\cdot)$ analogous to §6.7.1, system (6.7.12) in the mild form is described by

$$z(t) = z_0 + z_0^1 \frac{t^2}{2} + \ldots + z_0^k \frac{t^k}{k!} + L(t)Nz(t) + L(t)Bu$$

where $B = I$ (identity on $U$) again and $L(\cdot)$ is now given by

$$L(t)z(\cdot) = \int_0^t \int_0^{s_{k-1}} \ldots \int_0^{s_1} z(ds)ds\ ds_1 \ldots ds_{k-1} .$$

If $k$ is odd, then $LBe_{nm} = \Phi_{nm}$, where

$$\Phi_{nm} = \begin{cases} ( \quad t^k \quad , T^k)\ \psi_n & n \in \mathbb{N} , m = 0 \\ (\lambda_k T^k/(2m\pi)^k . \sin(2m\pi/T)t , 0 )\ \psi_n & n \in \mathbb{N} , m = 1,2,\ldots \\ (\lambda_k T^k/(2m\pi)^k . \cos(2m\pi/T)t , 0 )\ \psi_n & n \in \mathbb{N} , m = -1,-2,\ldots \end{cases}$$

and $\lambda_k = (-1)^{(k-1)/2}$. If $k$ is even then $LBe_{mn} = \Phi_{n,-m}$ for $\Phi_{nm}$ defined above with $\lambda_k = (-1)^{k/2}$.

The results here follow analogous to the previous paragraph. Here too (6.7.8) holds for any $s$ (see Remark A in §6.7.1), and (6.7.10) holds for $Z' = Z$ (c.f. Remark C in §6.7.1). So we can make $U$ as smooth as we like. Also note that (6.7.11) is satisfied for any nonlinearity

$$z(\cdot) \mapsto f\left(\frac{d^k z}{dt^k}, \ldots, \frac{dz}{dt}, z(\cdot), t\right)$$

which is continuous from $H^s(0,T;Z)$ to the larger space $H^{s-k}(0,T;Z)$ . (c.f. Remark E.)

6.7.3 - Third example:

Consider the system

$$\dot{z}(t) = Az(t) + Nz(t) + u(t) \quad , \qquad z(0) = z_0 \tag{6.7.13}$$

where $Z$ is a Hilbert space, $A:\mathcal{D}(A) \to Z$ is a self-adjoint operator with eigenvalues $\{\lambda_n\}_{n\in\Gamma}$ for some countable set $\Gamma$ and $u \in \mathcal{F}(0,T;Z)$ (see page 8 for definition of $\mathcal{F}(D,Z)$). Clearly if $Z$ is a finite dimensional space then $\dim(Z) = \#\Gamma$ whereas if $Z$ has infinite dimension then $\Gamma$ will have infinite elements (e.g., $\Gamma = \mathbb{N}$ ) .

If $\{\psi_n\}_{n\in\Gamma}$ are the corresponding eigenvectors then, from the expansion

$$z_0 = \sum_{n\in\Gamma} \langle z_0,\psi_n\rangle_Z \, \psi_n$$

we have that for $z_0 \in \mathcal{D}(A)$ , $Az$ is given by

$$Az_0 = \sum_{n\in\Gamma} \lambda_n \langle z_0,\psi_n\rangle_Z \, \psi_n \, .$$

Also, if $\{\lambda_n\}_{n\in\Gamma}$ are bounded from above (i.e., if $\exists \lambda \in \mathbb{R}$ such that $\lambda_n \leq \lambda$ for all $n \in \Gamma$) then $A$ generates a strongly continuous semi-group $S(\cdot): \mathbb{R} \to \mathcal{L}(Z)$ which is given by

$$S(t)z_0 = \sum_{n\in\Gamma} e^{\lambda_n t} \langle z_0,\psi_n\rangle_Z \, \psi_n \, .$$

System (6.7.13) in the mild form becomes

$$z(t) = S(t)z_0 + L(t)Nz(t) + L(t)Bu \qquad z(0) = z_0 \tag{6.7.14}$$

where the operator $L(\cdot)$ is defined by (2.2.11), namely

$$L(t)z(\cdot) = \int_0^t S(t-s)\; z(s)ds \;,$$

and $B = I$ (identity on $U$) . Set $X = M^s(0,T;Z)$ for some $s$ and take the space of input functions $U = U'(e_{nm}, \alpha_{nm}, \Delta)$ , where $\Delta = \Gamma \times \mathbb{Z}^+$ ,

$$e_{nm}(t) = a_m\, t^m\, \psi_n \qquad \text{for all} \quad (n,m) \in \Delta$$

and $a_m > 0$ are constants which we can choose freely. For example, we can set $a_m = 1$ for all $(n,m) \in \Delta$ or, instead, $a_{nm} = 1/T^m$ , in which case

$$e_{nm}(T) = \psi_n \qquad \text{for all} \quad (n,m) \in \Delta \;.$$

Alternatively we could set $a_m = [(2m+1)/T^{2m+1}]^{\frac{1}{2}}$ , in which case

$$\| e_{nm} \|_{L^2(0,T;Z)} = 1 \qquad \text{for all} \quad (n,m) \in \Delta \;,$$

or $a_m$ such that for some $s$

$$\| e_{nm} \|_{H^s(0,T;Z)} = 1 \qquad \text{for all} \quad (n,m) \in \Delta \;,$$

etc. So, the control action will be the functions $u$ which have the form

$$u(t) = \sum_{n\in\Gamma} (u_{n0} + u_{n1}t + u_{n2}t^2 + \ldots)\psi_n$$

with norm given by

$$\| u \|_U^2 = \sum_{n\in\Gamma} \sum_{m\in \mathbb{Z}^+} \left|\frac{u_{nm}}{a_m}\right|^2 . \alpha_{nm} ,$$

where the constants $\alpha_{nm}$ are still to be define. Since

$$\int t^m e^{\ t} dt = \frac{t^m e^{\lambda t}}{\lambda} - \frac{m}{\lambda} \int t^{m-1} e^{\lambda t} dt$$

$$= e^{\lambda t} \sum_{j=0}^{m} \frac{(-1)^{m-j} \ m!}{j! \ \lambda^{m-j+1}} \ t^j$$

we have that $L(\cdot)Be_{nm}$ is given by

$$L(t)Be_{nm} = \psi_n e^{\lambda_n t} \int_0^t e^{-\lambda_n s} a_m s^m ds \qquad \text{for all} \quad (n,m) \in \Delta$$

$$= \psi_n(p_{nm}(t) - q_{nm}(t)) \qquad \text{for all} \quad (n,m) \in \Delta \ ,$$

where $p_{nm}(t)$ is the polynomial of degree $m$ defined for each $(n,m) \in \Delta$ by

$$p_{nm}(t) = \sum_{j=0}^{m} c_{nmj} t^j$$

$$c_{nmj} = \frac{(-1)^{m-j} \ m! \ a_m}{j! \ (-\lambda_n)^{m-j+1}} \qquad \text{for all} \begin{cases} n \in \Gamma \\ m \in \mathbb{Z}^+ \\ 0 \le j \le m \end{cases}$$

and $q_{nm}(t)$ is given for each $(n,m) \in \Delta$ by

$$q_{nm}(t) = c_{nmo}\, e^{\lambda_n t} \quad .$$

Now set $r_{nm}(\cdot) = (p_{nm}(\cdot) - q_{nm}(\cdot))$ and $\bar{c}_{nm} = r_{nm}(T) = (p_{nm}(T) - q_{nm}(T))$ , then

$$L(t)Be_{nm} = r_{nm}(t)\psi_n \qquad \text{for all} \quad (n,m) \in \Delta$$

$$L(T)Be_{nm} = \bar{c}_{nm}\psi_n \qquad \text{for all} \quad (n,m) \in \Delta \quad .$$

It is clear that $\{L(\cdot)Be_{nm}\}_{(n,m)\in\Delta}$ is linearly independent (since $p_{nm}$ has degree m for each $(n,m) \in \Delta$ and $\{\psi_n\}_{n\in\Gamma}$ is a c.o.s.), and therefore,

$$R = \{LBe_{nm}\}_{(n,m)\in\Delta}$$

is linearly independent as well. For $(n,m) \quad \Delta$ set

$$\alpha_{nm} = \| LBe_{nm} \|^2_{M^s(0,T;Z)} = \| L(\cdot)Be_{nm} \|^2_{H^s(0,T;Z)} + \| L(T)Be_{nm} \|^2_Z$$

then we obtain, by theorem 5.9.3,

$$LB: U \to M^s(0,T;Z) \quad \text{is bounded and has closed range,}$$

that is, LB satisfies (6.7.2) when $\alpha_n$ is given as above.

Note that $\alpha_{nm}$ is well defined for each $(n,m) \in \Delta$ since $L(\cdot)Be_{nm} \in H^s(0,T;Z)$ and hence (6.7.8) holds. (See Remark A in §6.7.1.) Also observe that for $r \in \mathbb{R}$ , the range of

$$L(T)B : H^r(0,T;U) \to Z \quad , \qquad U = Z$$

is dense in $Z$ , i.e., $\mathcal{R}(L(T)B)^{\perp} = \{0\}$ , for any $r$ . However, it is possible that, for $r = 0$ $\mathcal{R}(L(T)B) \neq Z$ , i.e.,

$$L(T)B : L^2(0,T;Z) \to Z$$

does not have closed range (see Remark C in §6.7.1). In this case, although smoother spaces $X = \mathcal{M}^s$ make $\mathcal{U}$ smoother and smoother (in t), $\mathcal{U}$ will never be smoother than $Z$ (with respect to $U$). To overcome this problem we set a new state space $Z' \subset Z$ such that (6.7.10) holds, that is, for some $U$ (e.g. $U = Z$)

$$L(T)B : L^2(0,T;U) \to Z' \quad \text{has closed range.}$$

We shall then be able to make $\mathcal{U}$ as smooth (in t) as we like, (and not larger than $Z$ with respect to $U$) , by setting $\alpha_{nm}$

$$\alpha_{nm} = \| LBe_{nm} \|^2_{\mathcal{M}^s(0,T;Z')} = \| L(\cdot)Be_{nm} \|^2_{H^s(0,T;Z')} + \| L(T)Be_{nm} \|^2_{Z'}$$

for all $(n,m)$ $\Delta$ and $s$ large enough.

So, the adjustment of $\mathcal{U}$ and $X$ is actually a compromise we have to obtain between the spaces $\mathcal{U}$ , $Z'$ and $X = \mathcal{M}^s(0,T;Z')$. For example, consider the following particular case of the present example:

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z(x,t)}{\partial x^2} + Nz(x,t) + u(x,t)$$
$$z(0,t) = z(1,t) = 0 \quad , \quad z(x,0) = z_0(x) \tag{6.7.15}$$

For the above system (nonlinear difusion equation) we have

$$A = \frac{\partial}{\partial x^2} \ , \qquad \lambda_n = -n^2\pi^2 \quad , \qquad \psi_n(x) = \sqrt{2}\,\sin n\pi x \qquad \text{and} \qquad \Gamma = \mathbb{N} \ .$$

Usually the state space $Z$ taken for this system is $Z = L^2(0,1)$ . However, when the space of input functions is $L^2(0,T;Z)$ the system formed by the linear part of (6.7.15) is only approximate controllable to $Z$ (see page 58 of [11] ). That is, for the operator

$$L(T)B : L^2(0,T;L^2(0,1)) \to L^2(0,1) \ ,$$

$V = R(L(T)B)$ is such that $\bar{V} = Z$ but $V \neq Z$ (i.e., $V$ is smoother than $Z$) . In fact here we have that, for $\alpha_{nm}$ as defined above, $\mathcal{U}$ will become smoother and smoother (in t) as s increases but larger than $L^2(0,1)$ (with respect to U). A function $u(x,t) \in \mathcal{U}$ will be such that $u(x,\cdot)$ is smooth but $u(\cdot,t)$ may not be a $L^2(0,1)$ function. To overcome this problem we have to choose a new state space $Z'$ smoother than $L^2(0,1)$ (c.f. Remark C of §6.7.1).

If $Z' = H_0^1(0,1)$ then (see page 59 of [11]) the system formed by the linear part of (6.7.15) is exactly controllable to $Z$ when the controls $L^2(0,T;Z)$ are applied. That is, for the operator

$$L(T)B : L^2(0,T;L^2(0,1)) \to H_0^1(0,1) \ ,$$

$V = R(L(T)B) = Z'$ . So, if we set $U = L^2(0,1)$ , our new state space

$Z' = H_0^1(0,1)$ and redefine $\alpha_{nm}$ for all $(n,m) \in \Delta$ by

$$\alpha_{nm} = \| LBe_{nm} \|^2_{M^s(0,T;Z')} = \| L(\cdot)Be_{nm} \|^2_{H^s(0,T;Z')} + \| L(T)Be_{nm} \|^2_{Z'}$$

then

$LB: U \to X = M^s(0,T;Z')$ is bounded and has closed range,

$U$ becomes smoother and smoother (in t) as s increases, and not larger than $L^2(0,1)$ (with respect to U).

Here we make a remark: although it is true that by enlarging s $U$ becomes smoother and smoother, it is not totally true that $U$ can be as smooth as we like (e.g. $U \simeq H^1$, $H^2$, etc.). In fact, here $U$ cannot be strictly smoother than $L^2(0,T;Z)$. To understand the situation, define $U_m$ the subspace of $U$ which contains all the polynomials of order up to m, that is

$$U_m = \{u = \sum_{j=0}^{m} \sum_{n\in\Gamma} u_{nj} t^j \psi_n \in U\} .$$

So $U_0 \subset U_1 \subset U_2 \subset \dots$ and

$$U_m^{\perp} \{u = \sum_{j>m} \sum_{n\in\Gamma} u_{nj} t^j \psi_n \in U\} .$$

What really happens here is that for $s \geq 0$ $U$ is at least as smooth as $L^2(0,T;Z)$. If $s > m$, then $U_m$ is smoother than $L^2(0,T;Z)$ and

$U_m^{\perp}$ is as smooth as $L^2(0,T;Z)$ . So, by enlarging $s$ we enlarge the subspace $U_m \subset U$ which is smoother than $L^2(0,T;Z)$ .

So we have obtained a space $U$ and a new state space $Z'$ for which (6.7.2) holds with $X = M^s(0,T;Z')$ . Here we followed the procedure of §5.10.4. As in the previous examples, we did not have to construct a representation for $X$ , (as shown in §5.9.2), to determine $U$ . However, since such representation is useful for expressing operators such as $\Pi$, LB, $(LB)^{\dagger}$ , etc., we show it here: First note that

$$E_2 = E_4 = \emptyset \ .$$

Set

$$\Lambda_0 = \{(n,m)\in\Delta \ : \ \bar{c}_{nm} = 0\} \ , \qquad \Lambda_1 = \Delta \setminus \Lambda_0 \qquad \text{and} \qquad \Lambda_3 = \Lambda_1 \ .$$

(Observe that $\Lambda_0$ may be empty.) Now we can set $\Phi_{inm}$ , $i = 0,1,3$, $(n,m) \in \Delta$ , as defined in (5.9.8) for $R = \{LBe_{nm}\}_{(n,m)\in\Delta}$ . That is,

$$\begin{aligned}
\Phi_{0nm} &= (L(\cdot)B\,e_{nm} \ , \quad 0 \quad ) && (n,m) \in \Lambda_0 \\
\Phi_{1nm} &= (L(\cdot)B\,e_{nm} \ , \ L(T)B\,e_{nm}) && (n,m) \in \Lambda_1 \\
\Phi_{3nm} &= (L(\cdot)B\,e_{nm} \ , \quad 0 \quad ) && (n,m) \in \Lambda_3 = \Lambda_1 \ ,
\end{aligned}$$

and then we can write

$$X = M^s(0,T;Z') \simeq E_0 \oplus E_1 \oplus E_3$$

where

$$E_i = X'(\Phi_{inm}, \beta_{inm}, \Lambda_i) \qquad \text{for} \quad i = 0,1,3$$

and

$$\beta_{inm} = \| \Phi_{inm} \|^2_{M^2(0,T;Z')} \qquad \text{for all } (n,m) \in \Lambda_i \ , \quad i = 0,1,3.$$

Obviously if $\Lambda_0 = \emptyset \Rightarrow E_0 = \emptyset \Rightarrow X = E_1 \oplus E_3$ . Now, since

$$R(LB) = \overline{R(LB)} = E_0 \oplus E_1 \qquad \text{and} \qquad R(LB)^{\perp} = E_3 \ ,$$

we shall group the spaces $E_0$ and $E_1$ to give $X$ a more compact representation in the form

$$X \simeq R(LB) \oplus R(LB)^{\perp} \ .$$

Clearly

$$R(LB) = E_0 \oplus E_1 = X' \ (\Phi_{nm}, \alpha_{nm}, \Delta) \ ,$$

where

$$\Phi_{nm} = LBe_{nm} = \begin{cases} \Phi_{0nm} & \text{if} \quad (n,m) \in \Lambda_0 \\ \\ \Phi_{1nm} & \text{if} \quad (n,m) \in \Lambda_1 \ . \end{cases}$$

So, if $z \in X = M^S(0,T;Z')$ then $z$ can be expressed in the form

$$z = \sum_{(n,m)\in\Delta} a_{nm} \Phi_{nm} + \sum_{(n,m)\in\Lambda_3} a_{3nm}\Phi_{3nm}$$

and $\| \cdot \|_{M^S}$ is equivalent to the following norm

$$\| z \|^2 = \sum_{(n,m)\in\Delta} |a_{nm}|^2 \alpha_{nm} + \sum_{(n,m)\in\Lambda_3} |a_{3nm}|^2 \beta_{3nm} .$$

Now LB , $(LB)^{\dagger}$, L(T)B, $\Pi_1$, $\Pi_2$, $\Pi$, etc., all have simple expressions too. For example

$$\Pi z = \sum_{(n,m)\in\Delta} a_{nm} \Phi_{nm} \qquad \text{for any } z \in X$$

$$LBu = \sum_{(n,m)\in\Delta} u_{nm} \Phi_{nm} \qquad \text{for any } u \in \mathcal{U}$$

and

$$(LB)^{\dagger} z = \sum_{(n,m)\in\Delta} a_{nm} e_{nm} \qquad \text{for any } z \in X .$$

This simple representation is due to the fact that

$$M = (\{e_{nm}\}_{(n,m)\in\Delta} , \{\Phi_{nm}\}_{(n,m)\in\Delta} \cup \{\Phi_{nm}\}_{(n,m)\in\Lambda_3} ,\Delta,\Delta,\Delta\cup\Lambda_3)$$

is a complete matched set for the primitive operator $(LB)^1 : [\mathcal{U}] \to M^s(0,T;Z')$ of LB . (c.f. Remark B in §6.7.1.)

Suppose that $L = (L(\cdot),L(T))$ satisfies (2.2.13) and (2.2.14) for some space $\mathcal{Y}$ (see §2.2.3, page 13), then (6.7.11) holds for any non-linearity N which is continuous from $H^s(0,T;Z')$ to $\mathcal{Y}$ . (c.f. Remark E.) For instance, consider the system described by the nonlinear diffusion equation (6.7.15). If the state space $Z' = H_0^1(0,1)$ then since the semigroup $S(\cdot)$ satisfies

$$S(\cdot) \in L^1[0,T;\mathcal{L}(H^{-1+\delta}(0,1),H_0^1(0,1))]$$

for any small $\delta > 0$ , (6.7.11) is satisfied for nonlinearities $N$ which are continuous from $H_0^1(0,1)$ to the larger space $H^{-1+\delta}(0,1)$ . This allows for a large class of nonlinearities to be considered. For example $Nz = z^4$ , $(\frac{\partial z}{\partial x})^2$ , $z(\frac{\partial z}{\partial x})$ , $z^2(\frac{dz}{\partial x})$ , etc.. .

6.7.4 - Fourth example:

Consider the system

$$\dot{z}(t) = Az(t) + Nz(t) + bu(t) , \qquad z(0) = z_0 \tag{6.7.16}$$

where $Z$ is a Hilbert space, $A$ as is the previous paragraph, $b \in Z$ satisfy

$$b_n = \langle b,\psi_n\rangle_Z \neq 0 \qquad \text{for all} \quad n \in \Gamma ,$$

and the control $u(t)$ is a function from $[0,T]$ to $\mathbb{R}$ , that is $u \in \mathcal{F}(0,T;\mathbb{R})$ (see page 8 for definition of $\mathcal{F}(D,\mathbb{R})$).

Now let $i$ be any element in $\Gamma$ and set $\{\psi_n'\}_{n\in\Gamma}$

$$\psi_n' = \begin{cases} \psi_n & \text{if} \quad n \neq i , \quad n \in \Gamma \\ b & \text{if} \quad n = i \end{cases} . \tag{6.7.17}$$

That is, $\{\psi_n'\}_{n\in\Gamma} = \{b\} \cup \{\psi_n\}_{n\in\Gamma\setminus\{i\}}$ . Since $b_n \neq 0$ we have that

$$\text{Span}\{\psi_n'\}_{n\in\Gamma} = \text{Span}\{\psi_n\}_{n\in\Gamma} \tag{6.7.18}$$

and also $\overline{\mathrm{Span}\{\psi_n\}}_{n\in\Gamma} = \overline{\mathrm{Span}\{\psi_n'\}}_{n\in\Gamma} = Z$ .

System (6.7.16) can be expressed in the mild form (6.7.14) with $B:\mathcal{U} \to \mathcal{F}(0,T;Z)$ being

$$(Bu)(t) = bu(t) \qquad \text{for } u \in \mathcal{U}$$

(see Remark D of §6.7.1).

It has been shown in §3.13 of [11] that the system formed by the linear part of (6.7.16) ($\dot{z} = Az + bu$ , $z(0) = z_0$) is approximately controllable to $Z$ on $[0,T]$ when the input are the $L^2(0,T)$ functions. That is, the range of the operator

$$L(T)B:L^2(0,T) \to Z$$

is dense in $Z$ . Set $V = \mathcal{R}(L(T)B)$ , then $\bar{V} = Z$ . However, $V$ may be a smoother space than $Z$ (i.e. $V$ may be strictly contained in $Z$ ).

We shall construct here a space $\mathcal{U}$ of input functions $u(t)$ from $[0,T]$ to $\mathbb{R}$ . Similarly to the previous examples, $\mathcal{U}$ will be such that (6.7.2) holds, which implies that (see §5.10.5)

$$L(T)B : \mathcal{U} \to Z$$

has closed range. From the above explanation it is clear that if $V \subsetneq Z$ , (if $V$ is strictly contained in $Z$) , then $\mathcal{U}$ will be larger than $L^2(0,T)$ , no matter how smooth we make $X = M^s(0,T;Z)$ .

However, we shall be able to obtain smoother spaces $\mathcal{U}$ if we restrict the state space to some $Z' \subset Z$ such that

$$L(T)B: \to Z' \quad \text{has closed range.}$$

So, the adjustment is in fact of the three spaces $\mathcal{U}$ , $Z'$ and $X = M^s(0,T;Z')$ .

Let first $X = M^s(0,T;Z)$ for some $s \in \mathbb{R}$ and take the space of input functions $\mathcal{U} = \mathcal{U}'(e_m,\alpha_m,\Delta)$ where $\Delta = \mathbb{Z}^+$ ,

$$e_m(t) = a_m t^m \qquad m \in \Delta = \mathbb{Z}^+ ,$$

$a_m > 0$ being constants which we can choose freely and $\alpha_m$ are still to be defined. So, the control action will be the functions $u$ which have the polynomial form

$$u(t) = \sum_{m\in \mathbb{Z}^+} u_m t^m = u_0 + u_1 t + u_2 t^2 + \ldots$$

with norm given by

$$\|u\|_{\mathcal{U}}^2 = \sum_{m\in \mathbb{Z}^+} (|u_m|^2 / a_m^2).\alpha_m .$$

Let $p_{nm}$ , $q_{nm}$ , $r_{nm}$ and $\bar{c}_{nm}$ be as in the previous example. Then, for $m \in \Delta = \mathbb{Z}^+$

$$L(t)Be_m = \sum_{n\in\Gamma} b_n\psi_n \, e^{\lambda_n t} \int_0^t e^{-\lambda_n s} a_m s^m ds$$

$$= \sum_{n\in\Gamma} b_n\psi_n(p_{nm}(t) - q_{nm}(t))$$

$$= \sum_{n\in\Gamma} r_{nm}(t)b_n\psi_n$$

$$L(T)Be_m = \sum_{n\in\Gamma} \bar{c}_{nm}b_n\psi_n \quad .$$

Since degree of $p_{nm} = m$ for each $n \in \Gamma$ and $\{\psi_n\}_{n\in\Gamma}$ is a c.o.s. in $Z$, we have that

$$R = \{LBe_m\}_{m\in\Delta}$$

is linearly independent. Now set for each $m \in \Delta = \mathbb{Z}^+$

$$\alpha_m = \| LBe_m \|^2_{M^s(0,T;Z)} = \| L(\cdot)Be_m \|^2_{H^s(0,T;Z)} + \| L(T)Be_m \|^2_Z \ .$$

for all $m \in \Delta = \mathbb{Z}^+$ then, by theorem 5.9.3, LB satisfies (6.7.2), i.e.,

$$LB: U \to M^s(0,T;Z) \text{ is bounded and has closed range.}$$

Note that the following

$$LBe_m \in M^s(0,T;Z) \qquad \text{for all } m \in \Delta \tag{6.7.19}$$

must hold in order to $\alpha_m$ be well defined for each $m$ (c.f. Remark A of §6.7.1). If $s \le 1$ then (6.7.19) holds. This follows since

$be_m \in C^1(0,T;Z)$ , $m \in \Delta = \mathbb{Z}^+$ and therefore (see theorem 2.21 of [ ]) we have for $m \in \Delta$

$$L(\cdot)Be_m \in H^1(0,T;Z) \qquad \text{and}$$

$$\frac{dL}{dt}(t)Be_m = \mathcal{S}(t)be_m(0) + L(t)Be_m'$$

$$= \delta_{m,0}\, a_m \mathcal{S}(t)b + L(t)Be_m'$$

where

$$e_m'(t) = \frac{de_m(t)}{dt} = \begin{cases} m\, a_m t^{m-1} & \text{if } m \geq 1 \\ 0 & \text{if } m < 1 \end{cases}$$

and $\delta_{j,k}$ is the Kronecker delta (i.e., $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ if $j \neq k$) . If we assume

$$b \in \mathcal{D}(A)$$

(rather than $b \in Z$ initially assumed) then (6.7.19) holds for $s = 2$ . This follows since now $\mathcal{S}(\cdot)b \in H^1(0,T;Z)$ , $\frac{d}{dt}\mathcal{S}(t)b = \mathcal{S}(t)Ab$ , and therefore, since $e_m' \in C^1(0,T;Z)$ , we have for $m \in \Delta$

$$L(\cdot)Be_m \in H^2(0,T;Z)$$

$$\frac{d^2}{dt^2}L(t)Be_m = \delta_{m,0}a_m\mathcal{S}(t)Ab + \delta_{m,1}\, ma_m\mathcal{S}(t)b + L(t)Be_m''$$

where

$$e_m''(t) = \frac{d^2 e_m(t)}{dt^2} = \begin{cases} m(m-1)a_m t^{m-2} & \text{if } m \geq 2 \\ 0 & \text{if } m < 2 . \end{cases}$$

It is not difficult to obtain the generalization of this result: If $s$ is an integer $\geq 2$ then (6.7.19) holds as long as $b, Ab, \ldots, A^{s-2}b \in \mathcal{D}(A)$, which is equivalent to

$$A^{s-2}b \in \mathcal{D}(A) .$$

Clearly if $Z$ is finite dimensional or $A$ is a bounded operator (which will be the case if and only if $\{\lambda_n\}_{n\in\Gamma}$ is bounded), then $\mathcal{D}(A) = Z$ and thus (6.7.19) holds for any $s$ .

We illustrate the case $A$ is unbounded with the following example: consider the system described by the nonlinear diffusion equation

$$\begin{gathered} \frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial x^2} + Nz(x,t) + b(x)u(t) \\ z(0,t) = z(1,t) = 0 \qquad z(x,0) = z_0(x) \end{gathered} \tag{6.7.20}$$

where the state space $Z = L^2(0,1)$. Here we have again $A = \partial^2/\partial x^2$ . So, if $b \in L^2(0,1)$ then $\alpha_m$ is well defined for each $m \in \Delta$ for $s \leq 1$ . For $s = 2,3,\ldots$ $\alpha_m$ will be well defined for each $m \in \Delta$ if, for example, $b$ satisfies

$$b \in H^{2(s-1)}(0,1) \cap H_0^1(0,1) .$$

Now we observe that even when (6.7.19) holds for a large $s$ it is not clear whether our space of input (polynomials) functions $\mathcal{U}$ is smoother than $L^2(0,T)$ . However, if we consider a new state space $Z' \subset Z$ such that (see Remark C of §6.7.1),

$$L(T)B:L^2(0,T) \to Z'$$

has closed range, and redefine $\alpha_m$ for all $m \in \Delta = \mathbb{Z}^+$

$$\alpha_m = \| LBe_m \|^2_{\mathcal{M}^s(0,T;Z')} = \| L(\cdot)Be_m \|^2_{H^s(0,T;Z')} + \| L(T)Be_m \|^2_{Z'}$$

then we have that

$$LB:\mathcal{U} \to X = \mathcal{M}^s(0,T;Z') \quad \text{is bounded and has closed range,}$$

$\mathcal{U}$ becomes smoother and smoother as $s$ increases and if $s \geq 0$ $\mathcal{U}$ is certainly not larger than $L^2(0,T)$. What really happens is that the larger $s$ is the larger is the subspace of $\mathcal{U}$ which is smoother than $L^2(0,T)$ . We can represent this by setting $\mathcal{U}_m$ the subspace of $\mathcal{U}$ which contains all the polynomials of order up to $m$ ,

$$\mathcal{U}_m = \{u = \sum_{j=0}^{m} u_j t^j \in \mathcal{U}\}$$

which implies that $\mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \ldots$ and

$$\mathcal{U}_m^{\perp} = \{u = \sum_{j>m} u_j t^j \in \mathcal{U}\} \quad .$$

If $s > m$, then $\mathcal{U}_m$ is smoother than $L^2(0,T)$ and $\mathcal{U}_m^{\perp} \approx L^2(0,T)$. So, by increasing $s$ we enlarge the subspace $\mathcal{U}_m \subset \mathcal{U}$ which is smoother than $L^2(0,T)$.

So, we have obtained a space $\mathcal{U}$ and a new state space $Z'$ for which (6.7.2) holds for $X = M^s(0,T;Z')$. Observe that we did not have to construct any representation for $X$ to obtain $\mathcal{U}$. However, since such representation is useful we show it here: First note that $E_4 = \emptyset$. Now set

$$g_{nm}(t) = \psi_n' \, r_{nm}(t) \qquad \text{for all } n \in \Gamma,\ m \in \Delta$$

where $\psi_n'$ was defined in (6.7.17). Now, since $\psi_i = b$ for a fixed $i \in \Gamma$, and using (6.7.18), we have that (see section 5.4)

$$\{g_{nm}(\cdot)\}_{\substack{n\in\Gamma' \\ m\in\Delta}} \quad \text{is a completion for } \{LBe_m\}_{m\in\Delta} \text{ in } X,$$

where $\Gamma' = \Gamma \setminus \{\lambda_i\}$.

To ease the notation let $\eta\colon \mathbb{N} \to \Gamma \times \mathbb{Z}^+$ be any bijective mapping which sets

$$j \to (n_j, m_j) \in \Gamma \times \mathbb{Z}^+ = \Gamma \times \Delta \qquad \text{for each } j \in \mathbb{N}$$

and set $\Lambda_2 = \mathbb{N}$ and

$$f_j(\cdot) = g_{n_j m_j}(\cdot) \qquad \text{for each } j \in \Lambda_2$$

Also set

$$\Lambda_0 = \{m\in\Delta : L(T)Be_m = 0\} , \qquad \Lambda_1 = \Delta \setminus \Lambda_0 \qquad \text{and} \qquad \Lambda_3 = \Lambda_1 .$$

(Observe that $\Lambda_0$ may be empty.) Now we can set $\Phi_{im}$ , $i = 0,1,2,3, m \in \Delta$ , as defined in (5.9.8) for $R = \{LBe_m\}_{m\in\Delta}$ . That is

$$\begin{array}{lll} \Phi_{0m} = (L(\cdot)Be_m , & 0 & ) \qquad m \in \Lambda_0 \\ \Phi_{1m} = (L(\cdot)Be_m , & L(T)Be_m & ) \qquad m \in \Lambda_1 \\ \Phi_{2m} = ( f_m(\cdot) , & 0 & ) \qquad m \in \Lambda_2 \\ \Phi_{3m} = (L(\cdot)Be_m , & 0 & ) \qquad m \quad \Lambda_3 , \end{array}$$

and then we can write

$$X = M^S(0,T;Z') \simeq E_0 \oplus E_1 \oplus E_2 \oplus E_3$$

where

$$E_i = X'(\Phi_{im} , \beta_{im} , \Lambda_i) \qquad \text{for } i = 0,1,2,3$$

and

$$\beta_{im} = \| \Phi_{im} \|^2_{M^S(0,T;Z')} \qquad \text{for all } m \in \Lambda_i , \quad i = 0,1,2,3.$$

If $\Lambda_0 = \emptyset \Rightarrow E_0 = \emptyset$ and $\quad = E_1 \oplus E_2 \oplus E_3$ . In any case, since

$$\Lambda_0 \cap \Lambda_1 = \Delta \qquad \text{and} \qquad \Lambda_0 \cap \Lambda_1 = \emptyset$$

we can group $E_0$ and $E_1$ to give an even simpler representation. Let $E = E_0 \oplus E_1 = X'(\Phi_m,\alpha_m,\Delta)$ where

$$\Phi_m = LBe_m = \begin{cases} \Phi_{0m} & \text{if } m \in \Lambda_0 \\ \Phi_{1m} & \text{if } m \in \Lambda_1 . \end{cases}$$

Then

$$X \simeq E \oplus E_2 \oplus E_3 ,$$

that is, if $z \in X = M^S(0,T;Z')$ then $z$ can be expressed as

$$z = \sum_{m\in\Delta} a_m \Phi_m + \sum_{i=2,3} a_{im} \Phi_{im}$$

and $\|\cdot\|_{M^S}$ is equivalent to the following norm

$$\|z\|^2 = \sum_{m\in\Delta} |a_m|^2 \alpha_m + \sum_{i=2,3} |a_{im}|^2 \beta_{im} .$$

Moreover,

$$\Pi z = \sum_{m\in\Delta} a_m \Phi_m \qquad \text{for any } z \in X ,$$

$$LBu = \sum_{m\in\Delta} u_m \Phi_m \qquad \text{for any } u \in U ,$$

and

$$(LB)^{\dagger} z = \sum_{m\in\Delta} a_m e_m \qquad \text{for any } z \in X .$$

As for the nonlinear part LN , we have again (as in the previous example) that if $L$ satisfies (2.2.13) and (2.2.14) for some space $\mathcal{Y}$, then (6.7.11) holds for any continuous nonlinearity $N$ from $H^s(0,T;Z')$ to $\mathcal{Y}$. (c.f. Remark E of §6.7.1.)

6.7.5 - Fifth example:

Consider the system described by the nonlinear wave equation

$$\frac{\partial^2 w(x,t)}{\partial t^2} = \frac{\partial^2 w(x,t)}{\partial x^2} + \mathcal{N}w(x,t) + u(x,t) \tag{6.7.21}$$

$$w(0,t) = w(1,0) = 0 , \qquad w(x,0) = w_0(x) , \qquad \dot{w}(x,0) = y_0(x)$$

If we set $y = \dot{w}$ and $A$ the operator from $\mathcal{D}(A) \subset L^2(0,1)$ to $L^2(0,1)$

$$Aw = -\frac{\partial^2 w}{\partial x^2} \qquad \mathcal{D}(A) = H^2(0,1) \cap H_0^1(0,1)$$

then we can write (6.7.21) in the form (3.2.3), namely

$$\dot{z} = \mathcal{A}z + Nz + \mathcal{B}u , \qquad z(0) = z_0 \tag{6.7.22}$$

where $z = (w,y)$ , $z_0 = (w_0,y_0)$

$$\mathcal{A}z = \mathcal{A}\binom{w}{y} = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix} \binom{w}{y}$$

$$Nz = N\binom{w}{y} = \binom{0}{\mathcal{N}w} \qquad \text{and} \qquad \mathcal{B}u = \binom{0}{I}u .$$

If we set the state space $Z$

$$Z = \mathcal{D}(A^{\frac{1}{2}}) \times L^2(0,1) = H_0^1(0,1) \times L^2(0,1)$$

$$\| z \|_Z^2 = \| (w,y) \|_Z^2 = \| w \|^2_{H_0^1(0,1)} + \| y \|^2_{L^2(0,1)}$$

then the operator $A:\mathcal{D}(A) \subset Z \to Z$ defined above generates the strongly continuous semigroup $S(\cdot)$ given by (see [11])

$$S(t)z = S(t)\binom{w}{y} = \begin{bmatrix} \sum_{n\in\mathbb{N}} \psi_n( \quad w_n \cos n\pi t + \frac{1}{n\pi} y_n \sin n\pi t) \\ \sum_{n\in\mathbb{N}} \psi_n(-n\pi\, w_n \sin n\pi t + \quad y_n \cos n\pi t) \end{bmatrix}$$

where $\psi_n(x) = \sqrt{2} \sin n\pi x$ , $n \in \mathbb{N}$ ,

$w_n = \langle w,\psi_n \rangle_{L^2(0,1)}$ and $y_n = \langle y,\psi_n \rangle_{L^2(0,1)}$ for all $n \in \mathbb{N}$ .

So, we can write (6.7.22) in the mild form

$$\begin{aligned} z(t) &= \begin{pmatrix} w(t) \\ y(t) \end{pmatrix} = S(t)\begin{pmatrix} w_0 \\ y_0 \end{pmatrix} + L(t)\begin{pmatrix} 0 \\ Nz \end{pmatrix} + L(t)Bu \\ z(0) &= (w(0),y(0)) = (w_0,y_0) \end{aligned} \tag{6.7.23}$$

where the operator $L(\cdot)$ is again defined by (2.2.11) and B from the space of input functions $\mathcal{U}$ to $\mathcal{F}(0,T;Z)$ as defined in section 3.1, i.e., $(Bu)(t) = \mathcal{B}u(t)$ .

It has been shown in §3.12 of [11] that the system formed by the linear part of (6.7.21) with input functions in $L^2(0,T;Z)$ is exactly controllable to $Z$ on $[0,T]$ for any $T > 0$. In other words, the range of

$$L(T)B : L^2(0,T;Z) \to Z$$

is $Z$. This will allow us to obtain spaces of input functions such that (6.7.2) holds for $X = M^s(0,T;Z)$ (i.e., without having to define a new state space $Z'$ and $U$ will become smoother if we increase $s$, though $U$ may never be strictly contained in $L^2(0,T;Z)$. See Remark E in §6.7.1.

Let $X = M^s(0,T;Z)$ for some $s \in \mathbb{R}$ and $U = U'(e_{nm}, \alpha_{nm}, \Delta)$ where here $\Delta = \mathbb{N} \times \mathbb{Z}^+$,

$$e_{nm}(t) = a_m t^m \psi_n \qquad \text{for all} \quad (n,m) \in \Delta$$

and $a_m > 0$ are constants, which we can choose freely. So, the control action is formed by the input functions $u$ which have the form

$$u(t) = \sum_{n \in \mathbb{N}} (u_{n0} + u_{n1} t + u_{n2} t^2 + \ldots) \psi_n$$

with norm defined by

$$\| u \|_U^2 = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{Z}^+} (|u_{nm}| / a_{mn})^2 . \alpha_{nm}$$

where the constant $\alpha_{nm}$ are still to be defined. Now,

$$L(t)Be_{nm} = \begin{bmatrix} \psi_n \frac{1}{n\pi} \int_0^t a_m s^m \sin n\pi(t-s)ds \\ \\ \psi_n \int_0^t a_m s^m \cos n\pi(t-s)ds \end{bmatrix}$$

$$= \begin{bmatrix} \psi_n \ (p_{nm}(t) - c_{nm} \cos n\pi t + \frac{d_{nm}}{n\pi} \sin n\pi t) \\ \\ \psi_n \ (p'_{nm}(t) + n\pi c_{nm} \sin n\pi t + d_{nm} \cos n\pi t) \end{bmatrix}$$

where $p_{nm}(t)$ is the polynomial of degree $m$ defined for each $(n,m) \in \Delta = \mathbb{N} \times \mathbb{Z}^+$ by

$$p_{nm}(t) = \sum_{j=0}^{[\frac{1}{2}m]} \tilde{c}_{nmj} \ t^{m-2j}$$

$$\tilde{c}_{nmj} = \frac{(-1)^j \ m! \ a_m}{(m-2j)! \ (-n\pi)^{2j+2}} \qquad \text{for all} \begin{cases} (n,m) \in \Delta \\ 0 \le j \le [\frac{m}{2}] \end{cases}$$

($[r]$ means greatest integer $\le r$ ; e.g. $[3\frac{1}{2}] = 3$ , $[\frac{1}{2}] = 0$ , etc.), $p'_{nm}(t)$ is the polynomial of degree $(m-1)$ defined for each $(n,m) \in \Delta$ by

$$p'_{nm}(t) = \frac{d}{dt} p_{nm}(t) = \begin{cases} \sum_{j=0}^{[\frac{1}{2}(m-1)]} (m-2j)\tilde{c}_{nmj} t^{m-2j-1} & \text{if } m \neq 0 \\ \\ 0 & \text{if } m = 0 \end{cases}$$

$$c_{nm} = \tilde{c}_{n,m,[\frac{1}{2}m]} \ \delta_{m,2.[\frac{1}{2}m]} \qquad (n,m) \in \Delta$$

$$d_{nm} = \tilde{c}_{n,m,[\frac{1}{2}(m-1)]} \ \delta_{(m-1),2.[\frac{1}{2}(m-1)]} \qquad (n,m) \in \Delta$$

($\delta_{j,k}$ represents the Kronecker delta, $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ if $j \neq k$). Now set

$$r_{nm}(t) = p_{nm}(t) - c_{nm} \cos n\pi t + \frac{d_{nm}}{n\pi} \sin n\pi t$$

$$r'_{nm}(t) = \frac{d}{dt} r_{nm}(t) = p'_{nm}(t) + n\pi \sin n\pi t + d_{nm} \cos n\pi t$$

$$\bar{r}_{nm} = r_{nm}(T) \qquad \text{and} \qquad \bar{r}'_{nm} = r'_{nm}(T) \quad .$$

Then

$$L(t)Be_{nm} = \begin{pmatrix} r_{nm}(t) \\ \\ r'_{nm}(t) \end{pmatrix} \psi_n \qquad L(T)Be_{nm} = \begin{pmatrix} \bar{r}_{nm} \\ \\ \bar{r}'_{nm} \end{pmatrix} \psi_n$$

for all $(n,m) \in \Delta = \mathbb{N} \times \mathbb{Z}^+$ .

Since $\{\psi_n\}_{n \in \mathbb{N}}$ is a c.o.s. in $L^2(0,1)$ , degree of $p_{nm} = m$ and for $m \neq 0$ , degree of $p'_{nm} = (m-1)$ for each $n \in \mathbb{N}$ , it is not difficult to verify that

$$R = \{LBe_{nm}\}_{(n,m) \in \Delta}$$

is linearly independent. Now set

$$\alpha_{nm} = \| LBe_{nm} \|^2_{M^s(0,T;Z)} = \| L(\cdot)Be_{nm} \|^2_{H^s(0,T;Z)} + \| L(T)Be_{nm} \|^2_Z$$

for each $(n,m) \in \Delta$ . By theorem 5.9.3 again, LB satisfies (6.7.2), i.e.,

$$LB: \mathcal{U} \to X = M^s(0,T;Z) \text{ is bounded and has closed range.}$$

Note that, since $L(\cdot)Be_{nm} \in H^s(0,T;Z)$ , $LBe_{nm} \in M^s(0,T;Z)$ and hence $\alpha_{nm}$ is well defined for each $(n,m) \in \Delta$ (c.f. Remark A in §6.7.1).

By making $X = M^s$ smoother, i.e. increasing $s$ , $\alpha_{nm}$ increases and hence $U$ becomes smoother. The larger $s$ is the smoother $U$ becomes. However, $U$ can never be strictly smoother than $L^2(0,T;Z)$ . This is because, similarly to the two last examples, there will always be a subspace of $U$ (which contains the polynomials of order $\geq s$ only) which will be as smooth as $L^2(0,T;Z)$ . That is, for $s \geq 0$ $U$ is at least $\simeq L^2(0,T;Z)$ and if $s > m$ the subspace $U_m$ of $U$ (which contains the polynomials of order up to $m$) ,

$$U_m = \{u = \sum_{j=0}^{m} \sum_{n\in\Gamma} u_{nj} t^j \psi_n \in U\}$$

is certainly smoother than $L^2(0,T;Z)$ .

So, here we have obtained the adjustment of $U$ and $X = M^s(0,T;Z)$ without having to redefine the state space $Z$ . Also, we did not have to construct a representation for $X = M^s$ (as in §5.9.2), though such representation would help to represent $LB$ , $L(T)B$ , $\Pi$, etc.

The nonlinear part of (6.7.21) will satisfy (6.7.11), that is (see Remark E of §6.7.1)

$$L\binom{0}{N} : H^s(0,T;H_0^1(0,1) \times L^2(0,1)) \to M^s(0,T;H_0^1(0,1) \times L^2(0,1))$$

is continuous, for a large class of nonlinearities $N$ . e.g. $Nw = w^2$ ,

$Nw = \cos w$ , $Nw = w \cos w$ , $Nw = a \sin w$ (which appears in Gordon's wave equation [ 41 ] , $Nw = -|w|^{\rho} w$ (which appears in applications of quantum mechanics [ 46]), etc.

6.7.6 - Sixth example:

Consider the hyperbolic system

$$\frac{\partial^2 w(x,t)}{\partial t^2} = \frac{\partial^2 w(x,t)}{\partial x^2} + Nw(x,t) + b(x)u(t)$$

$$w(0,t) = w(1,t) = 0 , \quad w(x,0) = w_0(x) , \quad \dot{w}(x,0) = y_0(x) \tag{6.7.24}$$

where $b \in L^2(0,1)$ and satisfy

$$b_n = \int_0^T b(x)\, \psi_n(x) dx = \langle b,\psi_n \rangle_{L^2(0,1)} \neq 0 \qquad n \in \mathbb{N}$$

for $\psi_n(x) = \sqrt{2} \sin n\pi x$ , $n \in \mathbb{N}$ ; and $u$ is a real valued function. If we set $y = \dot{w}$ , $z = (w,y)$, $z_0 = (w_0,y_0)$ , the operators $A$, $\mathcal{A}$, and $N$ and the state space $Z$ as in the previous example we can write system (6.7.24) in the form (6.7.22) where $\mathcal{B}: \mathbb{R} \to Z$ is defined as follows

$$\mathcal{B}u = \binom{0}{b} u ,$$

and in the mild form (6.7.23) where $B: \mathcal{U} \to X$ is

$$(Bu)(t) = \binom{0}{bu(t)} .$$

It has been shown by D.L. Russel in [47], that the system formed by the linear part of (6.7.21) is exactly controllable to

$$\mathcal{D}(A) = \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) = H^2(0,1) \cap H_0^1(0,1) \times H_0^1(0,1)$$

on $[0,T]$ if $T \geq 2$, the input of the system is the $L^2(0,T)$ functions and $b_{\cdot}$ satisfy the additional condition

$$\liminf_{n\to\infty} n.|b_n| > 0 . \tag{6.7.25}$$

Russel obtained this result by expanding the state $(w(\cdot),y(\cdot)) \in Z = H_0^1(0,1) \times L^2(0,1)$ in series of orthonormal functions and then showing the existence of solutions of (6.7.24) which are $0$ at $t = T > 2$ for any initial state $(w_0(\cdot),y_0(\cdot)) \in \mathcal{D}(A)$. Equivalently, $L(T)B$ regarded as an operator from $L^2(0,T)$ to $\mathcal{D}(A)$, i.e.,

$$L(T)B : L^2(0,T) \to \mathcal{D}(A) = H^2(0,1) \cap H_0^1(0,1) \times L^2(0,1)$$

satisfies

$$\mathcal{R}(L(T)B) = \mathcal{D}(A) .$$

We remark that in this thesis we also use expansion in series (for both the control $u \in \mathcal{U}$ and the state $z \in Z$), but in a different way to Russel in his work. The framework constructed in the theory developed in chapter 5 allows us to expand in linearly independent functions (such as $1,t,t^2,\ldots$) rather than orthonormal functions only. Also, with the results of section 5.7 we can give the right topology to the spaces $\mathcal{U}$ and $X$ (of these expanded functions) in order to certain properties for an operator from $\mathcal{U}$ to $X$ (e.g. $LB:\mathcal{U} \to X$) to satisfy. Moreover, our approach of the pair $z = (z(\cdot), z_f)$ of trajectory-final state is unique.

Since we already know that $\mathcal{R}(L(T)B) = \mathcal{D}(A)$ when the input functions are $L^2(0,T)$, it is reasonable to set the new state space $Z'$

$$Z' = \mathcal{D}(A) = H^2(0,1) \cap H_0^1(0,1) \quad \times \quad H_0^1(0,1)$$

so that the space of input functions $\mathcal{U}$ here will not be larger than $L^2(0,T)$.

Let $X = \mathcal{M}^s(0,T;Z')$ for some $s \in \mathbb{R}$ and $\mathcal{U} = \mathcal{U}'(e_m, \alpha_m, \Delta)$ where now $\Delta = \mathbb{Z}^+$,

$$e_m(t) = a_m t^m \qquad\qquad m \in \Delta = \mathbb{Z}^+$$

and $a_m > 0$ are constants which we can choose freely. So, the input functions $u$ are polynomials

$$u(t) = u_0 + u_1 t + u_2 t^2 + \dots$$

normed by

$$\| u \|_{\mathcal{U}}^2 = \sum_{n \in \mathbb{N}} (|u_m| / a_m)^2 \alpha_m .$$

Let $r_{nm}$, $r'_{nm}$, $\bar{r}_{nm}$, $\bar{r}'_m$ be as in the previous example. Then, for $m \in \Delta = \mathbb{Z}^+$, we have

$$LBe_m = \sum_{n \in \mathbb{N}} b_n \psi_n \begin{pmatrix} r_{nm}(\cdot) \ , \ \bar{r}_{nm} \\ \\ r'_{nm}(\cdot) \ , \ \bar{r}'_{nm} \end{pmatrix} .$$

It is easy to verify that

$$R = \{LBe_m\}_{m \in \Delta}$$

is a linearly independent set. So, if we set

$$\alpha_m = \| LBe_m \|^2_{M^s(0,T;Z')} = \| L(\cdot)Be_m \|^2_{H^s(0,T;Z')} + \| L(T)Be_m \|^2_{Z'}$$

for all $m \in \Delta = \mathbb{Z}^+$, we have that, by theorem 5.9.3,

$LB: \mathcal{U} \to M^s(0,T;Z')$ is bounded and has closed range.

If $s = 0$ $\alpha_m$ becomes

$$\alpha_m = \| LBe_m \|^2_{M^0(0,T;Z')} = \| L(\cdot)Be_m \|^2_{L^2(0,T;Z')} + \| L(T)Be_m \|^2_{Z'}$$

then (since the set $\{\| e_m \|^2_{L^2(0,T)} / \alpha_m\}_{m \in \mathbb{Z}^+}$ is bounded) we have that

$$\mathcal{U} \simeq L^2(0,T) .$$

However, for $s > 0$ $\alpha_m$ may be not well defined (see Remark A of §6.7.1). We would have to impose additional conditions to $b$ (such as $b \in \mathcal{D}(A^{\frac{1}{2}})$) similarly to example 6.7.4 in order to condition (6.7.8) be satisfied (i.e., in order to $\alpha_m$ be well defined) for $s > 0$. Unfortunately, condition (6.7.25) does not allow us to do that here.

For example, condition (6.7.25) and

$$b \in \mathcal{D}(A^{\frac{1}{2}}) = H_0^1(0,1)$$

cannot hold together.

Unlike the previous examples, here we cannot make $\mathcal{U}$ smoother and smoother because we cannot set $X = \mathcal{M}^s$ for large values of $s$ .

So, here we have obtained the adjustment of $\mathcal{U}$ , $Z'$ and $X = \mathcal{M}^0(0,T;Z')$ for which (6.1.2) is satisfied. Observe that we did not have to construct a representation for $X = \mathcal{M}^s$ to obtain the adjustment. However, such representation could be obtained (see §5.9.2) and would give simple expressions for operators such as $LB$ , $L(T)B$ , the projection $\Pi$ , etc.

## 6.8 - SOME APPLICATIONS.

In this section we show some examples which illustrate how one can apply the results of this chapter to solve the problem of nonlinear controllability as posed in §6.1.1. We shall see the utility of our approach which considers the pair $(z(\cdot),z_f)$ trajectory-final state and the projections which were used to construct the mappings $F$ in section 6.4.

First we consider the system (defined as in (3.2.3) with $\mathcal{B} = I$)

$$\dot{z}(t) = Az(t) + Nz(t) + u(t) \quad , \qquad z(0) = z_0 \ . \tag{6.8.1}$$

This system includes system (6.7.3) of §6.7.1, system (6.7.12) of §6.7.2, and system (6.7.13) of §6.7.3. It also includes, as a particular case, the nonlinear diffusion equation (6.7.15), namely

$$z_t = z_{xx} + Nz + u$$

$$z(0,t) = z(1,t) = 0 \quad , \qquad z(x,0) = z_0(x)$$

which has consistently been used as example in every previous paper which considered controllability via fixed point theorems (see examples in §3 of [ 9 ], §6 of [ 25], §5 of [35 ] and also in §1 of [45 ]).

We shall assume that the space of input functions $\mathcal{U}$ , a possibly new state space $Z' \subseteq Z$ and $X$ , a space which contains the pair $z = (z(\cdot), z_f)$ trajectory-final state (e.g., $X = M^S(0,T;Z')$) have been adjusted such that (5.10.4) holds, that is

$$LB:\mathcal{U} \to X \quad \text{has closed range,}$$

and that $\mathcal{U}_{ad} = \mathcal{U}$ . We have seen in section 5.10 two procedures to adjust these spaces. Also, unlike the papers mentioned above, where $\mathcal{U}_{ad}$ could only be taken $\mathcal{U}_{ad} = L^2(0,T,L^2(0,1))$ or even larger spaces containing distributions (as it is the case in [35 ]), we have seen in the last section that it is possible to take $\mathcal{U}_{ad}$ very smooth by restricting $X$ to smoother spaces. Moreover, we do not need to construct a representation for $X$ (as shown in §5.9.2) in order to make the adjustment, though of course it would help to express LB , L(T)B , the projections $\Pi$, $\bar{P}'$ , etc.

Observe that the theory developed in chapter 5 played an important role in both the adjustment and the representation of $X$. In the last section we present some specific examples.

The only condition we assume for the nonlinearity is (6.7.11), namely (see Remark E in §5.7.1).

$$LN: H^s(0,T;Z') \to X \quad \text{is continuous.} \tag{6.8.2}$$

So, unlike those papers mentioned, here we only require continuity. We do *not* impose any additional condition on the nonlinearity, (such as $N$ satisfies Lipschitz-type condition, or $N$ is compact, etc.).

We also admit

$$N(0) \neq 0 ,$$

a relaxation which was not allowed in any of those papers mentioned. Moreover, we also admit

$$z(0) \neq 0 ,$$

another relaxation which would upset the treatment of every one of those papers mentioned.

Note that here we can talk about controllability to the origin, a case which does not make much sense in the approach of the previous papers where the initial state $z_0$ was always assumed to be zero.

From the above explanation it is clear that the following theorem represents a considerable generalization of results from [9 , 25 , 35 and 45]. Observe that the proof involves results from chapter 4 (projections), chapter 5 (matched sets theory) and the present chapter 6.

6.8.1 - Theorem:

Let $\mathcal{U}_{ad} = \mathcal{U}$, $Z' \subseteq Z$ and $X$ be adjusted such that $\mathcal{R}(LB)$ is closed, $\Pi$ be the projection onto $\mathcal{R}(LB)$ defined by (4.3.12), $A$ and N as defined in (3.2.3), $Sz_0 \in X$ (see Remark F in §6.7.1) and $\mathcal{B} = I$.

If the linear system

$$\dot{z}(t) = Az(t) + \mathcal{B}u(t) , \qquad z(0) = z_0$$

is approximately controllable to $Z$ on $[0,T]$ (i.e., if $\mathcal{R}(L(T)B)^{\perp} = \{0\}$) and (6.8.2) holds (i.e., LN is continuous), then system (6.8.1), namely

$$\dot{z}(t) = Az(t) + Nz(t) + \mathcal{B}u(t) , \qquad z(0) = z_0$$

is exactly controllable to $Z'$ on $[0,T]$ (or equivalently; approximately controllable to $Z$ on $[0,T]$).

Moreover, for any $z_d \in Z'$ let

$$\tilde{z} = (\tilde{z}(\cdot), z_d) \in X$$

for any chosen trajectory $\tilde{z}(\cdot)$ (not necessarily such that $\tilde{z}(T) = z_d$) and $z^* = (z^*(\cdot), z_f^*)$ be given by

$$z^* = \Pi\tilde{z} + (I - \Pi)Sz_0 .$$

Then the control $u^* \in \mathcal{U}_{ad}$ defined by

$$u^* = (LB)^{\dagger} (\tilde{z} - LNz^* - Sz_0)$$

drives the system from $z_0$ at $t = 0$ to $z_d$ at $t = T$ .

Furthermore,

$$z_f^* = z_d$$

and $z^* = (z^*(\cdot), z_d)$ is the actual pair trajectory-final state when the control $u^*$ is applied.

Remarks:

i) Observe that $\mathcal{R}(\Pi) \oplus \mathcal{R}(I - \Pi) = X$ and

$$z^* = z_1 + z_2$$

where $z_1 = \Pi\tilde{z} \in \mathcal{R}(\Pi) = \mathcal{R}(LB)$ and $z_2 = (I - \Pi)Sz_0 \in \mathcal{R}(I - \Pi)$ . If we give $X$ the topology where $\Pi$ is an orthogonal projection (see §5.9.2), then

$z_1$ is the closest trajectory to $\tilde{z}$ in $\mathcal{R}(LB)$ and

$z_2$ is the closest trajectory to $Sz_0$ in $\mathcal{R}(LB)^{\perp}$ .

ii) If, for the controls $L^2(0,T;U)$ , $U = Z$ , the linear system is not exactly controllable to $Z$ and we take $Z' = Z$ then $\mathcal{U}$ will be a larger space than $L^2(0,T;U)$ with respect to $U$ (and possibly in $t$ as well). If however we take $Z'$ such that the linear system is exactly controllable to $Z'$ then $\mathcal{U}$ will be as smooth as (if not smoother) than $L^2(0,T;U)$ when $X$ is taken smooth enough.

Proof (of theorem 6.8.1):

First we see that, by (6.8.2), $LNz(\cdot) \in X$ for each $(z(\cdot),z_f) \in X$ and since $\Pi$ is a projection onto $S = L\mathcal{B}(\mathcal{U}_{ad}) = \mathcal{R}(L)$ ,

$$\Pi LNz(\cdot) = LNz(\cdot) \tag{6.8.3}$$

for all trajectories $z(\cdot)$ such that $(z(\cdot),z_f) \in X$ .

Now observe that, by (5.10.14), since $\mathcal{R}(L(T)B)^{\perp} = \{0\}$ ,

$$\Pi(\tilde{z}(\cdot),z_d) = (L(.)Bu \ , \ z_d) \qquad \text{for some} \quad u \in \mathcal{U}$$

and

$$(I-\Pi)Sz_0 \ = (\bar{z}(\cdot) \ , \ 0\ ) \tag{6.8.4}$$

for some $\bar{z}(\cdot)$ such that $(\bar{z}(\cdot),0) \in X$ . Hence,

$$z^* = (z^*(\cdot),z_f^*) = (L(\cdot)Bu + \bar{z}(\cdot) \ , \ z_d)$$

and therefore $z_f^* = z_d$ . Now, for $\xi:X \to X$ defined by (6.1.7), we have

$$\begin{aligned}\xi(z^*) &= z^* - Sz_0 - LNz^*(\cdot) \\ &= \Pi\tilde{z} - \Pi Sz_0 - LNz^*(\cdot) \ .\end{aligned} \tag{6.8.5}$$

Thus, by (6.8.3)

$$\Pi\xi(z^*) = \xi(z^*) = z^* - Sz_0 - LNz^*(\cdot) \ . \tag{6.8.6}$$

Now we see that $z^* = (z^*(\cdot),z_f^*) = (z^*(\cdot),z_d)$ is a wanted state if and only if $z^*$ is a fixed point of some mapping $F$ in the family defined by (6.4.9). Set $\mathcal{P} = \Pi$ , then

$$F(z^*) = \mathcal{S}z_0 + LNz^*(\cdot) + \Pi\xi(z^*)$$

and by (6.8.6) we have that $z^* = F(z^*)$ . Finally, by (6.8.5), we can write $u^*$ as

$$u^* = (LB)^\dagger \xi(z^*)$$

and this implies that $u^*$ is a wanted control for which $z^*$ is the pair trajectory-final state (see §6.1.1). This concludes the proof.

Q.E.D.

6.8.2 - Corollary:

Let $V = \mathcal{R}(L(T)B)$ (see §5.10.5), $\mathcal{A}$ and $\mathcal{B}$ as in theorem 6.8.1. If the linear system

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t) , \qquad z(0) = z_0$$

is *not* approximately controllable to $Z$ but

$$\mathcal{S}(T)z_0 \in V$$

(which obviously includes the case $z_0 = 0$), then the results of theorem 6.8.1 still hold.

Proof:

Since $\mathcal{S}(T)z_0 \in V$ , (6.8.4) still holds and the result here follows analogous to theorem 6.8.1.

Next we consider systems of the type (3.2.3)

$$\dot{z}(t) = Az(t) + Nz(t) + \mathcal{B}u(t) , \qquad z(0) = z_0 \tag{6.8.7}$$

where $A:\mathcal{D}(A) \subset Z \to Z$ generates a strongly continuous semigroup on the Hilbert space

$$Z = Z_1 \times Z_2$$

and the operators $\mathcal{B}$ and $N$ can be expressed formally as

$$\mathcal{B} = \binom{0}{I} \qquad \text{and} \qquad N\binom{z'}{z''} = \binom{0}{\mathcal{N}z'}$$

for some nonlinear operator $\mathcal{N}:Z' \to \mathcal{F}(D',Z')$ , $D' = [0,T]$ .

This system includes, as a particular case, the nonlinear wave equation (6.7.21), namely

$$w_{tt} = w_{xx} + \mathcal{N}w + u$$

$$w(0,t) = w(1,t) = 0$$

$$w(x,0) = w_0(x) , \qquad \dot{w}(x,0) = y_0(x)$$

which, incidentally, was used as example in [45 ].

We shall assume again that the space of input functions $\mathcal{U}$ and $\mathcal{X}$ (of the type $\mathcal{M}(0,T;Z)$) have been adjusted such that (5.10.4) holds, i.e.,

$LB:\mathcal{U} \to \mathcal{X}$ has closed range and $\mathcal{U}_{ad} = \mathcal{U}$ . We have seen in section 5.10 (and in example 6.7.5) that the adjustment can be made such that $\mathcal{U}_{ad}$ is smoother and smoother, by restricting $X$ to even smoother spaces. We can also give a representation to $X$ (as shown in §5.9.2) in which LB , L(T)B , the projections $\Pi$ , $\bar{P}'$ , etc., will have simple expressions. However, such representation is not necessary for the above adjustment.

The only condition assumed for the nonlinearity is (6.7.11) again, namely (see Remark E in §6.7.1).

$$LN:H^s(0,T;Z') \to X \quad \text{is continuous.} \tag{6.8.8}$$

That is, no additional condition (such as N satisfies a Lipschitz-type condition or, N is compact, etc.) is assumed for N .

We also admit the following relaxations

$$N(0) \quad \text{is possibly} \neq 0$$

and

$$z(0) = z_0 \quad \text{is possibly} \neq 0 \ \text{too.}$$

So, here we can study the case of controllability to the origin.

None of the above generalizations were possible in the previous papers which treated nonlinear controllability via fixed point theorems since their approach did not consider the pair trajectory-final state.

6.8.3 - <u>Theorem</u>:

Let $\mathcal{U}_{ad} = \mathcal{U}$ , $Z' \subseteq Z$ and $X$ be adjusted such that $R(LB)$ is closed, $\Pi$ be as defined by (4.3.12), $Sz_0 \in X$ and $A,B$ and N as defined above.

If the linear system

$$\dot{z}(t) = Az(t) + Bu(t) , \qquad z(0) = z_0 \tag{6.8.9}$$

is approximately controllable to Z on [0,T] (i.e., if $R(L(T)B)^{\perp} = \{0\}$) and (6.8.8) holds (i.e., LN is continuous), then system (6.8.7), namely

$$\dot{z}(t) = Az(t) + Nz(t) + Bu(t) \quad , \qquad z(0) = z_0$$

is exactly controllable to Z' on [0,T] (or equivalently, approximately controllable to Z on [0,T]).

Moreover, for any $z_d \in Z'$ if we take

$$\tilde{z} = (\tilde{z}(\cdot), z_d) \in X$$

for any chosen trajectory $\tilde{z}(\cdot)$ (not necessarily such that $\tilde{z}(T) = z_d$) and $z^* = (z^*(\cdot), z_f^*)$ given by

$$z^* = \Pi\tilde{z} - (I - \Pi)Sz_0 ,$$

then the control $u^* \in U_{ad}$ given by

$$u^* = (LB)^{\dagger}(\tilde{z} - LNz^*(\cdot) - Sz_0)$$

steers the system from $z_0$ at $t = 0$ to $z_d$ at $t = T$ . Furthermore,

$$z_f^* = z_d$$

and $z^* = (z^*(\cdot), z_d)$ is the actual pair trajectory-final state when the control $u^*$ is applied.

Proof:

The proof follows analogous to theorem 6.8.1 since (6.8.3) holds here too, that is, for $(z(\cdot), z_f) \in X$

$$\Pi LNz(\cdot) = LNz(\cdot) \quad .$$

Q.E.D.

6.8.4 - Corollary:

Let $V = \mathcal{R}(L(T)B)$ . The results of theorem 6.8.1 still hold when the linear system (6.8.9) is not approximately controllable to $Z$ if

$$\mathcal{S}(T)z_0 \in V \ .$$

Proof:

Since $\mathcal{S}(T)z_0 \in V$ , (6.8.4) still holds and the result here follows analogous to theorem 6.8.1.

Next we consider systems of the type (3.2.3)

$$\dot{z}(t) = Az(t) + Nz(t) + \mathcal{B}u(t) \ , \qquad z(0) = 0 \tag{6.8.10}$$

for more general types of operator $\mathcal{B}$ . We shall assume again that the space of input functions $\mathcal{U}$ , $Z'$ and $X = M^S(0,T;Z')$ have been adjusted such that (5.10.4) holds, i.e.,

$$LB:\mathcal{U} \to X \text{ has closed range.}$$

We have already discussed the fact that a nonlinearity $N$ can map a space $H^s(0,T;Z')$ into a larger space $\mathcal{Y}$ and yet

$$L\bar{N}:H^s(0,T;Z') \to \mathcal{M}^s(0,T;Z') \text{ be continuous} \tag{6.9.11}$$

(see §2.2.3, §2.2.7 and Remark C of §6.7.1). This is due to the smoothing action of the semigroup $\mathcal{S}(\cdot)$ and the integration (i.e., the convolution $\int_0^t \mathcal{S}(t-s)Nz(s)ds$ which makes $L$ map $\mathcal{Y}$ into $\mathcal{M}^s$ continuously. Also, if $N$ satisfies the Lipschitz-type condition (2.2.6), namely

$$\| Nz(\cdot) - Nz'(\cdot) \|_{\mathcal{Y}} \le k(\| z(\cdot) \|_{H^s}, \| z'(\cdot) \|_{H^s}) \cdot \| z(\cdot) - z'(\cdot) \|_{H^s}$$

for all $z(\cdot), z'(\cdot) \in D' = B_a(0)$ in $H^s$ and $L:\mathcal{Y} \to \mathcal{M}^s$ satisfies (2.2.13) and (2.2.14), namely,

$$\| L(\cdot)y(\cdot) \|_{H^s} \le k_1' \| y(\cdot) \|_{\mathcal{Y}}$$

$$\| L(T)y(\cdot) \|_{Z} \le k_2' \| y(\cdot) \|_{\mathcal{Y}}$$

then (see §2.2.7) $LN:H^s \to \mathcal{M}^s$ satisfies (2.2.28), namely

$$\| LNz(\cdot) - LNz'(\cdot) \|_{\mathcal{M}^s} \le c \| x(\cdot) - x'(\cdot) \|_{H^s} \tag{6.8.12}$$

for all $z(\cdot)$, $z'(\cdot) \in D' = B_a(0)$ in $H^s$, where $c = \sqrt{c_1^2 + c_2^2}$ was defined in §2.2.7.

The above Lipschitz-type condition for the nonlinearity has been used in [9, 25, 35]. In order to draw comparisons between the results of those papers and here we shall also assume that

$$N(0) = 0 \qquad \text{and} \qquad z(0) = z_0 = 0 .$$

Also, similarly to [35], let $\ell' > 0$ be such that

$$\| (L(T)B)^{\dagger} \|_{\mathcal{L}(U,X)} \leq \ell' . \tag{6.8.13}$$

If the adjustment is made, as exemplified in section 6.7, with the space $U$ of the type

$$U = U'(e_j, \alpha_j, \Delta)$$
$$= \{u = \sum_{j \in \Delta} u_j e_j \quad : \quad \| u \|^2 = \sum_{j \in \Delta} |u_j|^2 \alpha_j\}$$

where $e_j(t)$ are linearly independent functions from $[0,T]$ to $U$, and $\alpha_j$ given by

$$\alpha_j = \| LBe_j \|^2_{M^s} = \| L(\cdot)Be_j \|^2_{H^s} + \| L(T)Be_j \|^2_{Z'}$$

then we have that

$$\| LBu \|^2_{M^s} \leq \sum_{j \in \Delta} \| u_j LBe_j \|^2_{M^s} = \sum_{j \in \Delta} |u_j|^2 \alpha_j = \| u \|^2_{U} .$$

So $\| LB \| = 1$ and by (6.8.13) we have that for $\ell = 1 + \ell'$

$$\| (I - \Pi_1) \|_{\mathcal{L}(X)} \leq \ell$$

where $\Pi_1$ is the projection defined in (4.3.9) (see example 4.3.6).

6.8.5 - Theorem:

Let the system (6.8.10) be exactly controllable to $Z'$, the spaces $\mathcal{U}$, $Z'$ and $X = M^s(0,T;Z')$ be adjusted such that (5.10.4) holds and the nonlinearity satisfy the above Lipschitz-type condition on $D'$. Define

$$K = \ell c \qquad \text{and} \qquad r = \frac{a(1-K)}{\ell'} .$$

If $K < 1$ and $z_d \in D'' = B_r(0)$ in $Z'$, then the mapping $F: X \to X$

$$F(z) = LNz(\cdot) + \Pi\xi(z(\cdot), z_d) \tag{6.8.14}$$

has at least one fixed point $z^* \in D' \times D''$. Moreover, if $z^*$ is any fixed point of $F$ then $z^*$ is a wanted control, that is, the control $u^*$ given by

$$u^* = (LB)^{\dagger}\xi(z^*)$$

drives the system to $z_d$ at $t = T$.

Proof:

First observe that $F$ in (6.8.14) is the same mapping $F$ given by (6.4.16) with $P = \Pi$ and $P_d : Z' \to Z'$ the projection $z_f \mapsto z_d$ onto the set $\{z_d\}$ . Thus, since $\Pi$ is a uniform projection, any fixed point of $F$ is a wanted state.

To show the existence of a fixed point $z^*$ of $F$ consider first the mapping $\bar{F} : X \to X$ .

$$\bar{F}(z) = \bar{z} + (I - \Pi_1)LNz(\cdot) \tag{6.8.15}$$

for $z = (z(\cdot), z_f) \in X$ , where $\bar{z}$ is given by

$$\bar{z} = \Pi_1(0, z_d) .$$

We have that

$$\begin{aligned} \| \bar{F}(z) - \bar{F}(z') \|_X &\le \ell \| LNz(\cdot) - LNz'(\cdot) \|_X \\ &\le K \| z(\cdot) - z'(\cdot) \|_{H^s} \\ &\le K \| z - z' \|_X \end{aligned}$$

for all $z, z' \in D' \times Z'$ . Now, since $K < 1$ and $F(0) = \bar{z}$ , we have that (see Contraction Mapping Principle, p.114 of [26]) $\bar{F}$ has a fixed point $z^*$ if the ball

$$B_{K/(1-K)}(\bar{z}) \text{ in } X$$

is contained in $D' \times Z'$ . This will be the case if

$$(1 + K/(1-K))\|\bar{z}\|_X \le a$$

and this is guaranteed if

$$\| z_d \|_{Z'} \le a(1-K)/\ell' \ .$$

So $\bar{F}$ has a fixed point $z^*$. Now we show that any fixed point $z^*$ of $\bar{F}$ is also a fixed point of $F$. Let $z^* = \bar{F}(z^*)$, then

$$\begin{aligned} z^* &= \bar{z} + (I-\Pi_1)LNz^*(\cdot) - \Pi_2 z^* + \Pi_2 z^* \\ &= (\bar{z} - \Pi_2 z^*) + (I-\Pi)LNz^* \ . \end{aligned}$$

Now, since $\bar{z}$ and $\Pi_2 z^* \in \mathcal{R}(LB) = \mathcal{R}(\Pi)$, we have that

$$\Pi z^* = \bar{z} - \Pi_2 z^* \ .$$

Now, since the system is exactly controllable to $Z'$, $\mathcal{R}(L(T)B)^{\perp} = \{0\}$ and therefore

$$z^* = LNz^* + \Pi\xi(z^*(\cdot), z_d) = F(z^*) \ .$$

This concludes the proof. Q.E.D.

Remarks:

The conditions of the above theorem are more relaxed than in the corresponding theorems in the papers mentioned. However, the advantage we would like to point out is that here we can have smooth spaces $\mathcal{U}$ (unlike [35] where $\mathcal{U}$ had to be enlarged including distributions).

Now we consider another class of nonlinearities, compact operators. Suppose

$$LN: H^s(0,T;Z') \to M^s(0,T;Z') \text{ is completely continuous} \tag{6.8.16}$$

(see §2.2.1 for definition). Also note that, for $\bar{c}_1$ and $\bar{c}_2$ defined in §2.2.3, LN satisfies

$$\| LNz(\cdot) \|_{M^s} \leq \bar{c} \, \| z(\cdot) \|_{H^s} \tag{6.8.17}$$

for all $z(\cdot) \in D' = B_a(0)$ in $H^s$ where $\bar{c} = \sqrt{\bar{c}_1^2 + \bar{c}_2^2}$ (see §2.2.7). We shall also assume, for the sake of simplicity, that

$$N(0) = 0 \qquad \text{and} \qquad z(0) = z_0 = 0 \, .$$

6.8.6 - Theorem:

Let the linear part of the system (6.8.10) be exactly controllable to $Z'$, the spaces $\mathcal{U}$, $Z'$ and $X = M^s(0,T;Z')$ be adjusted such that (5.10.4) holds and the nonlinearity N satisfy the above compactness property. Define

$$K = \ell\bar{c} \qquad \text{and} \qquad r = \frac{a(1-K)}{\ell'} \, .$$

If $K < 1$ and $z_d \in D'' = B_r(0)$ in $Z'$ then the mapping $F: X \to X$

$$F(z) = LNz(\cdot) + \Pi\xi(z(\cdot), z_d)$$

has at least one fixed point $z^* \in D' \times D''$. Moreover, if $z^*$ is any fixed point of $F$, then the control $u^*$ given by

$$u^* = (LB)^{\dagger}\xi(z^*)$$

drives the system to $z_d$ at $t = T$.

Proof:

Consider the mapping $\bar{F}$ given by (6.8.15). We can show, similarly to the proof of theorem 6.8.5, that if $z^*$ is a fixed point of $\bar{F}$ then $z^*$ is also a fixed point of $F: X \to X$ given by (6.4.16) with $P = \Pi$ and $P_d: Z' \to Z'$ the projection $z_f \mapsto z_d$ onto $\{z_d\}$. Then, if there exists such $z^*$, it is a wanted state. So, we only have to show that such $z^*$ exists, that is, $\bar{F}$ has a fixed point. Clearly

$$\begin{aligned} \| \bar{F}(z) \|_X &\le \| \bar{z} \|_X + \ell \| LNz(\cdot) \|_X \\ &\le \ell' \| z_d \|_{Z'} + K \| z(\cdot) \|_{H^s} \end{aligned}$$

for all $z(\cdot) \in D'$. Thus, if $z = (z(\cdot), z_f) \in B_a(0)$ in $X$, $\| z(\cdot) \|_{H^s} \le a$ and therefore, for $z_d \in B_r(0)$ in $Z'$ we have that

$$\| \bar{F}(z) \|_X \le a .$$

So, $F$ maps $B_a(0)$ in $X$ into itself. Now, using Schauder fixed point theorem (see section 2.6), we have that, by the compactness condition, $\bar{F}$ has fixed point. Q.E.D.

The same remarks of §6.8.5 apply here.

Next we consider the case $\mathcal{U}_{ad} \neq \mathcal{U}$ . This is obviously a generalization which was not permitted in any of the previous work and only the approach of the pair trajectory-final state and the projections allow us to do this here. However, now the mappings F may not have necessary conditions (see §6.1.2) in cases such as

- $\mathcal{U}_{ad}$ is bounded.

Nevertheless, we may still have sufficient conditions for F , that is, any wanted state $z^*$ (and hence any wanted control $u^*$) can be achieved via the fixed points of F . (See §6.1.2.)

We have seen examples of projection onto a set S in chapter 4 and also in §5.10.6 for the case

$$S = LB(\mathcal{U}_{ad}) .$$

A more specific example: if

$$\mathcal{U}_{ad} = B_a(0) \text{ in } L^2(0,T;U) = \{u \in L^2(0,T;U) : \| u \| \leq a\}$$

then the projection

$$\mathcal{P} = (LB)P_{ad}(LB)^{\dagger}$$

($P_{ad}$ being a-radical retraction on $L^2(0,T;U)$)

is a continuous projection onto $S = LB(\mathcal{U}_{ad})$ and $\| \mathcal{P}z \| \leq \| z \|$ . Alternatively, since S is bounded, closed and convex, the projection

(see §4.1.6)

$$\mathcal{P}z = \text{the closest element to } z \text{ in } S$$

is also a continuous projection onto $S$ with $\|\mathcal{P}z\| \le \|z\|$ .

Similarly we could construct continuous projections $\mathcal{P}$ onto $S = LB(\mathcal{U}_{ad})$ with $\|\mathcal{P}z\| \le \|z\|$ for other types of $\mathcal{U}_{ad}$ . For example

$$\mathcal{U}_{ad} = \{u \in L^2(0,T;U) : \qquad u(t) \in U_{ad} \subset U \ \text{ a.e.}\}$$

$$\mathcal{P} = (LB)\, P_{ad}(LB)^{\dagger}$$

$$(P_{ad}\, u)(t) = \begin{cases} u(t) & \text{if } u(t) \in U_{ad} \\ 0 & \text{if } u(t) \notin U_{ad} \end{cases}$$

(assuming $0 \in U_{ad}$).

If $U_{ad}$ is bounded, closed and convex then so is $\mathcal{U}_{ad}$ and again we have the alternative of the projection (see §4.1.6)

$$\mathcal{P}z = \text{the closest element to } z \text{ in } S .$$

Another example:

$$\mathcal{U}_{ad} = \{u = \sum_k u_k e_k \in \mathcal{U} : \qquad |u_k| \le \rho_k\}$$

$$\mathcal{P} = (LB)P_{ad}(LB)^{\dagger}$$

$$P_{ad}u = \sum_k u_k' e_k \quad , \qquad u_k' = \begin{cases} u_k & \text{if } |u_k| \le \rho_k \\ \rho_k & \text{if } |u_k| > \rho_k \end{cases} .$$

If $\{\rho_k\}$ is bounded then so is $\mathcal{U}_{ad}$ and again we can use the projection (see §4.1.6)

$$\mathcal{P}z = \text{the closest element to } z \text{ in } S .$$

In the sequel we shall assume that $\mathcal{U}$, $Z'$ and $X = M^s(0,T;Z')$ have been adjusted, $\mathcal{U}_{ad} \subseteq \mathcal{U}$ and $\mathcal{P}$ is a continuous projection with

$$\| \mathcal{P}z \| \leq \| z \| .$$

Let us consider first the linear case, that is $N = 0$ ,

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t) , \qquad z(0) = z_0 . \tag{6.8.18}$$

6.8.7 - Theorem:

If $\mathcal{U}_{ad}$ is bounded, $\mathcal{S}z_0 \in X$ and $z_d \in Z'$ then the mapping

$$F(z) = \mathcal{S}z_0 + \mathcal{P}\xi(z(\cdot),z_d) \tag{6.8.19}$$

has at least one fixed point. Every fixed point $z^* = (z^*(\cdot),z_f^*)$ of $F$ which satisfies

$$z_f^* = z_d$$

is a wanted state, that is (see §6.1.1), the control $u^* \in \mathcal{U}_{ad}$ given by

$$u^* = (LB)^{\dagger}\xi(z^*)$$

drives system (6.8.18) from $z_0$ at $t = 0$ to $z_d$ at $t = T$ .

Moreover, every wanted control $u^*$ of the system can be achieved via the fixed points of $F$ by the above procedure.

Proof:

Clearly $F$ given by (6.8.19) is the same mapping $F$ given by (6.4.12) with $N = 0$ and $P_d$ the projection $z_f \mapsto z_d$ onto $\{z_d\}$. So we only have to show that $F$ has at least one fixed point. First,

$$\begin{aligned} \| F(z) - F(z') \| &\le \| \mathcal{P}\xi(z(\cdot),z_d) - \mathcal{P}\xi(z'(\cdot),z_d) \| \\ &\le \| (z(\cdot),z_d) - (z'(\cdot),z_d) \| \\ &\le \| z - z' \| . \end{aligned}$$

So, $F$ is a non-expansive mapping. Now observe that if $D$ is the set

$$D = \mathcal{S}z_0 + S$$

then $F$ maps $D$ into $D$. Thus, since $\mathcal{U}_{ad}$ is bounded, $D$ is bounded and therefore (see section 2.6) $F$ has a fixed point.

Q.E.D.

Remarks:

i) The boundedness condition on $\mathcal{U}_{ad}$ in the above theorem cannot be removed since we must have $D$ bounded to guarantee a fixed point of $F$.

If $F$ was a contraction $D$ could be unbounded. (See section 2.6.) However $F$ here is only non-expansive and therefore $D$ must be bounded (see section 2.6) for us to be able to prove existence of a fixed point of F . (A typical example is $f: \mathbb{R} \to \mathbb{R}$ , $f(x) = x+1$ , which is non-expansive, maps $\mathbb{R}$ into $\mathbb{R}$ but has no fixed points.)

ii) We have already discussed in §6.4.4 that we consider F to be a reasonable mapping to be used in the search of a wanted state $z^* = (z^*(\cdot), z_f^*)$, in spite of the fact that fixed points of F will have to satisfy the a posteriori test

$$z_f^* = z_d \; .$$

F has sufficient conditions, that is, every wanted state (and therefore every wanted control), if there exists any, can be reached via its fixed points. However, some states $z^*$ which happen to satisfy

$$\xi(z^*) = P\xi(z^*(\cdot), z_d) \; ,$$

if there exists any, will also be a fixed point of F and will be eliminated by the above test.

To illustrate take the extreme case where $\mathcal{U}_{ad}$ has only one admissible control $u_{ad}$ ,

$$\mathcal{U}_{ad} = \{u_{ad}\}$$

and $P$ onto $S = LB(\mathcal{U}_{ad})$ is the projection

$$Pz = LBu_{ad} \qquad \text{for all } z \; .$$

Then F becomes

$$F(z) = Sz_0 + LBu_{ad} \qquad \text{for all } z$$

which has one unique fixed point $z^*$ given by

$$z^* = \mathcal{S}z_0 + LBu_{ad} \; .$$

Here $z^*$ will be a wanted state if

$$z^*(T) = \mathcal{S}(T)z_0 + L(T)Bu_{ad} = z_d$$

(i.e., if $u_{ad}$ drives the system to $z_d$ on $[0,T]$). Otherwise $z^*$ fails the a posteriori test and we can conclude that there is no $u \in \mathcal{U}_{ad}$ which drives the system to $z_d$ on $[0,T]$.

Now let us consider the nonlinear case.

$$\dot{z}(t) = \mathcal{A}z(t) + Nz(t) + \mathcal{B}u(t) \quad , \qquad z(0) = z_0 \; .$$

Here we can have similar results to theorem 6.8.7 if the nonlinearity satisfies, for example, the Lipschitz-type condition (6.8.12) or, if the nonlinearity satisfies the compactness condition (6.8.16) and (6.8.17) for some $\bar{c} \leq 1$ . We illustrate with the following theorem.

6.8.8 - Theorem:

If $\mathcal{U}_{ad}$ is bounded, $z_d \in Z'$ , N satisfies (6.8.16) and for some $D' \subset X'$ bounded $Nz(\cdot) \in D'$ for all $z(\cdot)$, then the mapping

$$F(z) = \mathcal{S}z_0 + LNz(\cdot) + \mathcal{P}\xi(z(\cdot),z_d) \tag{6.8.20}$$

has at least one fixed point. Every fixed point $z^* = (z^*(\cdot),z_f^*)$ of F which satisfies

$$z_f^* = z_d$$

is a wanted state, that is, the control $u^* \in \mathcal{U}_{ad}$ given by

$$u^* = (LB)^{\dagger}\xi(z^*)$$

drives the system from $z_0$ at $t = 0$ to $z_d$ at $t = T$.

Moreover, every wanted control $u^*$ of the system can be achieved via the fixed points of $F$ by the above procedure.

Proof:

Clearly $F$ in (6.8.20) is the same mapping $F$ given by (6.4.12) with $P_d$ being the projection $z_f \mapsto z_d$ onto $\{z_d\}$. So we only need to show that $F$ has a fixed point. Set

$$F_1(z) = \mathcal{S}z_0 + LNz(\cdot) \qquad \text{and} \qquad F_2(z) = \mathcal{P}\xi(z(\cdot),z_d)$$

then $F_1$ is completely continuous, $F_2$ is non-expansive and if $D$ is the set

$$D = \mathcal{S}z_0 + \quad D' \quad + S \quad ;$$

then, since $D$ is bounded, the mapping $F = F_1 + F_2$ maps $D$ into $D$ and therefore (see section 2.6) $F$ has at least one fixed point.

Q.E.D.

6.8.9 - Final Remark:

We have also developed, in section 6.4, mappings $F$ which have necessary conditions, that is, no a posteriori test to be satisfied. However, when $\mathcal{U}_{ad}$ is bounded those mappings will not, in general, satisfy conditions on most fixed point theorems which require $F$ to map $D$ into $D$ for some bounded set $D$.

For example, consider $F$ given by

$$F(z) = (0,\ z_1 - z_d) + Sz_0 + LNz(\cdot) + \mathcal{P}g(z) \tag{6.8.21}$$

for all $z = (z(\cdot), z_1) \in \mathcal{X}$, where $g: \mathcal{X} \to \mathcal{X}$ is *any* continuous function. This mapping is similar to $F$ in (6.4.9) but more flexible since we can choose $g(\cdot)$ whereas in (6.4.9) we had $\xi(z(\cdot), z_d)$ in the place of $g(z)$. Similarly to §6.4.2 we can show that $F$ here in (6.8.21) has necessary conditions. (Unlike $F$ in (6.4.9) however, $F$ here in (6.8.21) does not have sufficient conditions.) Observe, if $z^* = (z^*(\cdot), z_1^*)$ is a fixed point of $F$ in (6.8.21), we have

$$(z^*(\cdot), z_1^*) = (0,\ z_1^* - z_d) + Sz_0 + LNz^*(\cdot) + \mathcal{P}g(z^*)$$

and thus, since $\mathcal{P}$ is a projection onto $LB(\mathcal{U}_{ad})$,

$$(z^*(\cdot), z_1^*) - (0,\ z_1^* - z_d) = Sz_0 + LNz^*(\cdot) + LBu^*$$

for some $u^* \in \mathcal{U}_{ad}$. Thus,

$$(z^*(\cdot), z_d) = Sz_0 + LNz^*(\cdot) + LBu^*$$

that is

$$\begin{aligned} z^*(\cdot) &= S(\cdot)z_0 + L(\cdot)Nz^*(\cdot) + L(\cdot)Bu^* \qquad \text{and} \\ z_d &= S(T)z_0 + L(T)Nz^*(\cdot) + L(T)Bu^* = z^*(T)\ . \end{aligned}$$

i.e., $(z^*(\cdot),z_d)$ is a wanted state. Summarizing, if we can find a fixed point $z^* = (z^*(\cdot),z_1^*)$ for the mapping $F$ in (6.8.21) above then

$$(z^*(\cdot),z_d) \text{ is a wanted state.}$$

Using the terminology of §6.4.2, $(z^*(\cdot),z_d) = f(z^*)$ where $f$ is the operative function

$$(z(\cdot),z_1) \mapsto f(z(\cdot),z_1) = (z(\cdot),z_d) \; .$$

So, if for a particular system and a particular choice of $g$ we can prove that $F$ in (6.8.21) has a fixed point this implies that the system is controllable to $z_d$ and the control $u^* \in \mathcal{U}_{ad}$ given by

$$u^* = (LB)^{\dagger}\xi(z^*(\cdot),z_d)$$

will drive the system from $z_0$ at $t = 0$ to $z_d$ at $t = T$ .

In general however it is very difficult to prove existence of a fixed point of $F$ in (6.8.21) because it is difficult to find a bounded set $D$ in $X$ for which $F$ maps $D$ into $D$ .

# CHAPTER 7.

# STATE ESTIMATION.

## 7.1 - INTRODUCTION.

### 7.1.1 - The problem of state estimation

• The problem of state estimation which we study here is: for a given output $y_e \in \mathcal{Y}$ and a given $\varepsilon \geq 0$ we want to determine whether there is a pair initial state-trajectory $(z_0, z(\cdot))$ such that $z = (z_0, z(\cdot))$ satisfies the dynamical equation (3.2.9) and the output equation (3.2.10) with a maximum error $\varepsilon$, that is

$$z = \mathcal{S}z_0 + LNz(\cdot) \qquad (7.1.1)$$

$$Cz(\cdot) \in \overline{B_\varepsilon(y_e)} . \qquad (7.1.2)$$

In the terminology of chapter 3 this is the same to say that $z$ is an $\varepsilon$-estimated state of the system. In particular, when we set $\varepsilon = 0$, (7.1.1) becomes the output equation

$$y_e = Cz(\cdot) \qquad (7.1.3)$$

and in this case $z$ will be an estimated state.

Let $\mathcal{X} = M^p(0,T;Z)$ where $p = 0$ or $1 \leq p < \infty$ (see §2.2.6), that is

$$\mathcal{X} = Z \times C(0,T;Z) \qquad \text{if } p = 0$$

or

$$\mathcal{X} = Z \times L^p(0,T;Z) \qquad \text{if } 1 \leq p < \infty$$

with $\|\cdot\|_{M^0}$ and $\|\cdot\|_{M^p}$ given in (2.2.24) and (2.2.25) respectively. Consider the subspace $S \subset X$ defined by (4.2.4), namely (see example 4.2.7)

$$S = R(\mathcal{S}) = \{z = (z_0, \mathcal{S}(\cdot)z_0) \in X : z_0 \in Z\} \ .$$

We have seen in example 4.2.7 that $\overline{P}:X \to X$ given by (4.2.3), namely

$$\overline{P}z = \overline{P}(z_0, z(\cdot)) = (z_0, \mathcal{S}(\cdot)z_0) \tag{7.1.4}$$

is a continuous projection onto $S$. By theorem 4.1.3 this implies that $S$ is closed and hence $\mathcal{S}^\dagger \in \mathcal{L}(X,Z)$. Therefore,

$$P = \mathcal{S}^\dagger \mathcal{S} \tag{7.1.5}$$

is the orthogonal projection onto $S$.

Clearly, by definition, $z = (z_0, z(\cdot)) \in X$ is an $\varepsilon$-estimated state if and only if

$$z - LNz(\cdot) \in S \tag{7.1.6}$$

and (7.1.2) hold.

Observe that the statement is also valid when we substitute (7.1.6) by

$$z(\cdot) - L(\cdot)Nz(\cdot) \in R(\mathcal{S}(\cdot)) \ .$$

However, in this case $R(\mathcal{S}(\cdot))$ would not necessarily be a closed subspace of $L^p(0,T;Z)$ (or $C(0,T;Z)$), whereas $S = \mathcal{R}(\mathcal{S})$ is always a closed subspace of $X = M^p(0,T;Z)$ . For this reason it will be more convenient to look at the pair $(z_0,z(\cdot))$ initial state-trajectory in a space $X = M^p(0,T;Z)$ rather than to the trajectory $z(\cdot)$ only. Also, this approach will bring some anologies, though not many, with the problem of $\varepsilon$-controllability studied in chapter 6 in the case $\mathcal{U}_{ad} = \mathcal{U}$ . Actually one can see this by comparing (7.1.6)-(7.1.2) with (6.1.4)-(6.1.5).

Let $\mathcal{P}$ and $P_e$ be mappings $\mathcal{P}:X \to X$ and $P_e:\mathcal{Y} \to \mathcal{Y}$

$$\mathcal{P} = \text{any continuous projection onto } S \text{ , and}$$

$$P_e = \text{any continuous projection onto } \overline{B_\varepsilon(y_e)} \text{ .}$$

The simplest example for $P_e$ is the translated $\varepsilon$-radial retraction (see example 4.1.5(i)). For $\mathcal{P}$ we have already seen the following examples: $\mathcal{P} = P$ in (7.1.5), $\mathcal{P} = \overline{P}$ in (7.1.4) and also $\mathcal{P} = \overline{P}'$ in (4.2.6) which is a uniform projection in the first component (see §4.7.1).

Many other examples of $\mathcal{P}$ could be defined and we consider here an interesting particular case. Suppose that

$$\mathcal{R}(C\mathcal{S}(\cdot)) \text{ is closed in } \mathcal{Y} \text{ .} \tag{7.1.7}$$

From the results of chapter 5 we know that if this is not the case it can be achieved by adjusting the spaces $Z$ and/or $\mathcal{Y}$ . In this case the mapping $\Pi_1:X \to X$

$$\Pi_1 z = \Pi_1(z_0,z(\cdot)) = \mathcal{S}(C\mathcal{S}(\cdot))^{\dagger}Cz(\cdot) \tag{7.1.8}$$

is a continuous projection onto the subspace $S_1$ given by

$$S_1 = \{(z_0, Sz_0) \in X : z_0 \in N(CS(\cdot))^{\perp}\} \subset S \ . \tag{7.1.9}$$

Now if we define $\Pi_2 : X \to X$ by

$$\Pi_2 = (I - \Pi_1)P$$

for any $P$ continuous projection onto $S$, then $\Pi : X \to X$

$$\Pi = \Pi_1 + \Pi_2 \tag{7.1.10}$$

is also a continuous projection onto $S$.

Observe that, in general, we do not require (7.1.7) to hold since we can obtain projections $P$ onto $S$ without having to impose this condition. However, in the particular case $P = \Pi$ assumption (7.1.7) is necessary.

We also define $\xi : X \to X$ by

$$\xi(z) = z - LNz(\cdot) \ , \quad z = (z_0, z(\cdot)) \in X \tag{7.1.11}$$

and hence (7.1.6) can be written as

$$\xi(z) \in S \ . \tag{7.1.12}$$

We denote

$E_s^{\varepsilon}$ = the set of the $\varepsilon$-estimated states,

and for $\varepsilon = 0$, $E_s^{\varepsilon}$ becomes

$E_s^0$ = the set of the estimated states.

7.1.2 - The mappings $F$ .

Similarly to chapter 6 we shall present here some mappings $F:X \to X$ together with an operative function $f:X \to X$ which in most of the cases is the identity on $X$ , (i.e., $f = I$) and information about elements in $f(\nu(F))$ may give us information about the $\varepsilon$-estimated states.

Ideally $F$ would be such that

$$E_s^\varepsilon = f(\nu(F))$$

which is the same as

$$x \in f(\nu(F)) \Longleftrightarrow x \in E_s^\varepsilon$$

that is, the mapping $F$ provides necessary and sufficient condition for us to obtain the $\varepsilon$-estimated states from its fixed points. This is not the case in general, specially for mappings $F$ obtained in earlier attempts to solve state estimation via fixed points. We say that $F$ has necessary conditions (to obtain the $\varepsilon$-estimated states) if

$$x \in f(\nu(F)) \Rightarrow x \in E_s^\varepsilon$$

which is equivalent to $f(\nu(F)) \subset E_s^\varepsilon$ . We say that $F$ has sufficient conditions (to obtain the $\varepsilon$-estimated states) if

$$x \in f(\nu(F)) \Leftarrow x \in E_s^\varepsilon$$

which is equivalent to $f(\nu(F)) \supset E_s^\varepsilon$ .

If necessity does not hold then an a posteriori test must be given in order to check whether each element of $f(\nu(F))$ is an $\varepsilon$-estimated state or not. Clearly the set

$$J_N(F) = f(\nu(F)) \setminus E_S^{\varepsilon}$$

represents the elements of $f(\nu(F))$ which will fail in the a posteriori test and the set

$$J_S(F) = E_S^{\varepsilon} \setminus f(\nu(F))$$

represents the $\varepsilon$-wanted states which cannot be obtained by the fixed points of $F$. If $F$ has necessary conditions then $J_N(F) = \emptyset$ and if $F$ has sufficient conditions, $J_S(F) = \emptyset$.

6.1.3 - <u>Desired properties of F</u>.

Firstly, we would like that

(a) $F$ has necessary conditions,

(b) $F$ has sufficient conditions.

We remember that the nicest feature of property (a) is that, when it holds we can determine existence of $\varepsilon$-wanted states by determining existence of fixed points of $F$.

We also want that

(c) $F$ includes the approach of $\varepsilon$-estimated states, $\varepsilon \geq 0$, rather than only the case of estimated states only, to allow the possibility of an error in the measurements.

Finally, consider the subspace $\mathcal{R}(CS(\cdot)) \subset \mathcal{Y}$ . We shall look for mappings F which satisfies

(d) F do not impose conditions on $\mathcal{R}(CS(\cdot))$ .

We shall see in the next section mappings $F = \phi$ which do not satisfy (d) because they impose condition (7.1.7) or, more restrictive than that,

$$\mathcal{R}(CS(\cdot)) = \mathcal{Y} \quad . \tag{7.1.13}$$

7.2 - HISTORICAL VIEW OF PREVIOUS WORK.

The first paper to use the approach of fixed points to state estimation, [8], came shortly after [25], the first paper to use fixed points in control. After this [9, 35 and 15] also dealt with this problem. We shall write the result of these papers using our terminology. In this section $f = I$ , so that the operative set

$$f(\nu(F)) = \nu(F) = \text{the fixed points of } F$$

and $\varepsilon = 0$ , since the approach of $\varepsilon$-estimated states were not considered in the above papers.

The first three papers, [8, 9 and 35] utilized mappings $\phi$ to obtain $z^*(\cdot)$ , an estimation of the trajectory of the system. In both [8] and [9] the mapping employed was

$$\phi z(\cdot) = S(\cdot)\, H^{-1}\, (y_e - CL(\cdot)Nz(\cdot)) + L(\cdot)Nz(\cdot) \qquad (7.2.1)$$

where $H = CS(\cdot)$ . It was assumed that the system formed by the linear part, that is

$$\dot{z}(\cdot) = S(\cdot)z_0$$

is continuously initially observable and this is equivalent to (7.1.13), i.e.,

$$\mathcal{R}(CS(\cdot)) = \mathcal{Y} \qquad (7.2.2)$$

and that

$$H^{-1}: \mathcal{Y} \to Z \ \text{ exists and is continuous.} \qquad (7.2.3)$$

Now observe that (7.2.2) => $(CS(\cdot))^{\dagger} = H^{\dagger} = H^{-1} \in \mathcal{L}(\mathcal{Y},Z)$ and hence $\Pi_1$ given by (7.1.8) is a continuous projection onto $S_1$ defined in (7.1.9). Moreover, (7.2.3) => $\mathcal{N}(CS(\cdot)) = \mathcal{N}(H) = \{0\}$ => $\mathcal{N}(CS(\cdot))^{\perp} = Z$ , and this implies that in this case

$$S_1 = S .$$

Define $\Phi: X \to X$

$$\Phi(z) = ((\phi z(\cdot))(0) \ , \ \phi z(\cdot)) \ . \qquad (7.2.4)$$

It is easy to verify that $\Phi$ can be expressed in the form of $F$ in (6.3.1) with

$$\eta = \xi \ , \quad P = \Pi_1 \quad , \quad S' = S'' = S_1 = S$$

and

$$Q(z) = SH^{-1}(y_e - Cz(\cdot)) .$$

Clearly $Q$ is active under $\Pi_1$ since $R(Q) \subset R(S) = S$ (see section 4.5). Applying theorem 6.3.1 we obtain

$$z^* = \Phi(z^*) \iff \begin{cases} \xi(z^*) = 0 \\ Q(z^*) = 0 \end{cases}$$

and since

$$Q(z^*) = 0 \iff y_e = Cz^*(\cdot)$$

we have that

$$z^* \in \nu(\Phi) \iff z^* \in E_S^0$$

thus $\Phi$ has necessary and sufficient conditions. So, the mapping $\phi$ satisfies both (a) and (b). This was achieved by conditions (7.2.2)-(7.2.3) i.e., continuous initial observability of the linear part. Summarizing, $\phi$ satisfies (a) and (b) but it does not satisfy (c) and (d).

In [35] A.J. Pritchard considered the mapping $\phi$ modified to

$$\phi z(\cdot) = SH^{\dagger}(y_e - CL(\cdot)Nz(\cdot)) + L(\cdot)Nz(\cdot) . \tag{7.2.5}$$

The assumption of continuous initial observability of the linear part was dropped and instead condition (7.1.7), namely

$$R(CS(\cdot)) \text{ is closed in } Y$$

was introduced. It was assumed in [35] that $\mathcal{Y}$ and/or $\mathcal{Z}$ could be adjusted in order to the operator $H = C\mathcal{S}(\cdot)$ have closed range so that the assumption holds. In the light of the results of chapter 5 we now know that this adjustment is possible.

Unfortunately $\phi$ in (7.2.5) do not have necessary conditions and the following a posteriori test had to be satisfied by $z^*(\cdot) \in \nu(\phi)$

$$(y_e - CL(\cdot)Nz^*(\cdot)) \in \mathcal{R}(C\mathcal{S}(\cdot)) \quad . \tag{7.2.6}$$

This condition was called "check of consistence" in [35].

If we define $\Phi$ as in (7.2.4) using $\phi$ in (7.2.5) then $\Phi$ can be written in the form of $F$ in (6.3.1) with

$$\eta = \xi \ , \quad P = \Pi_1 \ , \quad S' = S_1$$

and

$$Q(z) = -\mathcal{S}H^{\dagger}(y_e - Cz(\cdot)) \tag{7.2.7}$$

and again $Q$ is an active mapping under $\Pi_1$ since $\mathcal{R}(Q) \cap \mathcal{N}(\Pi_1) = \{0\}$ (see section 4.5). Setting $S'' = S$ and applying theorem 6.3.1 we obtain that (6.3.3) does not hold, which implies that $\Phi$ does not have sufficient conditions, but (6.3.2) does, that is

$$z^* = \Phi(z^*) \quad \Rightarrow \begin{cases} Q(z^*) = 0 \\ \xi(z^*) \in S \ . \end{cases}$$

But this implies that $\Phi$ does not have necessary conditions either, since $Q(z^*) = 0$ is not equivalent to (7.1.3). However, $Q(z^*) = 0$ together with the a posteriori test (7.2.6) do imply (7.1.3), that is

$$\left.\begin{array}{l} z^* \in \nu(\Phi) \\ \text{a posteriori test (7.2.6) satisfied} \end{array}\right\} \Rightarrow \quad z^* \in E_s^0 .$$

Summarizing, $\phi$ in (7.2.5) does not satisfy any of the properties (a)-(d).

In [15] the author introduced the spaces $M^p(0,T;Z)$ and an approach to the problem using linear projections. We presented in [15] the class of mappings $F:X \to X$ (parameterized the $P$, the projection used) given by

$$F(z) = \zeta + (I-P)LNz(\cdot) + Pz - SH^{\dagger}Cz(\cdot) \tag{7.2.8}$$

where $\zeta$ is a constant term given by

$$\zeta = SH^{\dagger}y_e .$$

It was assumed the same a posteriori test used in [35], that is, condition (7.2.6). It was shown in [15], and it is easy to verify, that when the projection $P$ used was $\Pi = \Pi_1+\Pi_2$ defined in (7.1.10), then $F$ could be expressed as

$$F(z) = ((\phi z(\cdot))(0),\phi z(\cdot)) + \Pi_2(z - LNz(\cdot))$$

where $\phi$ was given by (7.2.5). Moreover, $z^*(\cdot) \in \nu(\phi) \Rightarrow (z^*(0),z^*(\cdot)) \in \nu(F)$

whereas the converse is not true. So, the approach of $\phi$ in (7.2.5) was incorporated in the mappings $F$ .

Now observe that $F$ in (7.2.8) can be written as

$$F(z) = LNz(\cdot) + \mathcal{P}\xi(z) - Q(z) \tag{7.2.9}$$

where $Q$ is given by (7.2.7). Setting

$$\eta = \xi \ , \quad P = \mathcal{P} \ , \quad S' = S \ ,$$

then $Q$ is active under $P$ since $\mathcal{R}(Q) \subset \mathcal{R}(P)$ and we can apply theorem 6.3.1. By (6.3.3), since $z^* \in E_s^0 \Rightarrow \xi(z^*) \in S$ ,

$$z^* \in \nu(F) \quad \Leftarrow \quad z^* \in E_s^0$$

so $F$ has sufficient conditions. By (6.3.2),

$$z^* \in \nu(F) \quad \Rightarrow \begin{cases} Q(z^*) = 0 \\ \xi(z^*) \in S \end{cases}$$

and this implies that

$$\left.\begin{array}{l} z^* \in \nu(F) \\ \text{a posteriori test (7.2.6) satisfied} \end{array}\right\} \Rightarrow \quad z^* \in E_s^0 \ .$$

Observe that in general we do not have to assume any condition on $\mathcal{R}(CS(\cdot))$ and hence $F$ satisfies property (d). Summarizing $F$ satisfies properties (b) and (d) but it does not satisfy neither (a) nor (c).

The author has also shown in [15] that if (7.2.6) holds for all

$z(\cdot) \in L^p(0,T;Z)$ (or for all $z(\cdot) \in C(0,T;Z)$ in the case $p = 0$), then $F$ provides necessary and sufficient conditions. This includes cases where the system formed by the linear part is continuously initially observable. That is, $F$ generalizes $\phi$ in (7.2.1) too, when the same assumptions are imposed.

Remark.

The analysis of the mappings $\phi$ and $F$ in this section was simpler than a similar analysis made by the author in [15]. This was due to theorem (6.3.1).

The results of section 3.3 about overall observability were originally developed by the author to be applied when solving the joint problem of state and parameter estimation of a system of the type (3.3.1) via the state estimation of the overall system (3.3.5). Assumption of continuous initial observability required for the mapping $\phi$ in (7.2.1) becomes then continuous initial overall observability of system (3.3.3).

## 7.3 - NEW CLASSES OF MAPPINGS F .

We recall that, for §7.1.1, $z$ is an $\varepsilon$-estimated state if and only if

$$\xi(z) \in S = \mathcal{R}(S) \tag{7.3.1}$$

and

$$Cz(\cdot) \in \overline{B_\varepsilon(y_e)} \quad \text{in } \mathcal{Y} \tag{7.3.2}$$

where $\xi: \mathcal{X} \to \mathcal{X}$ is defined by (7.1.11), namely

$$\xi(z) = z - LNz(\cdot) \quad , \quad z = (z_0, z(\cdot)) \in X \, .$$

Also, $P$ and $P_e$ represent any continuous projections onto $S$ and $\overline{B_\varepsilon(y_e)}$ respectively. If $\varepsilon = 0$, (7.3.2) becomes the output equation

$$y_e = Cz(\cdot) \, . \tag{7.3.3}$$

In this section we also use the following notation

$$\mathfrak{F} = \begin{cases} L^p(0,T;Z) & \text{if } 1 \le p < \infty \\ C(0,T;Z) & \text{if } \quad p = 0 \, . \end{cases}$$

Hence,

$$X = Z \times \mathfrak{F} \, .$$

Now we consider new classes of mappings $F: X \to X$ to be used in the search of $\varepsilon$-estimated states.

7.3.1 - All properties (a)-(d) satisfied.

Consider the mapping $F$ in (7.2.9) with operative function $f = I$ and $Q$ a mapping of the type (2.2.4), namely

$$Q(z) = -q(z(\cdot))\bar{x}$$

where $\bar{x} \neq 0$, $\bar{x} \in S = R(S)$ is chosen arbitrarily and $q: \mathfrak{F} \to \mathbb{R}$ is any functional which satisfies

$$q(z(\cdot)) = 0 \quad \Longleftrightarrow \quad Cz(\cdot) \in \overline{B_\varepsilon(y_e)} \, . \tag{7.3.4}$$

Two examples of functionals $q$ satisfying (7.3.4) are

$$q(z(\cdot)) = \|P_e Cz(\cdot) - Cz(\cdot)\|_Y$$

and

$$q(z(\cdot)) = \frac{m\|P_e Cz(\cdot) - Cz(\cdot)\|_Y}{1 + \|P_e Cz(\cdot) - Cz(\cdot)\|_Y} \quad , \quad m \neq 0 \ . \tag{7.3.5}$$

In the particular case $\varepsilon = 0$ these two functionals have obvious particularizations, e.g., $q$ in (7.3.5) becomes

$$q(z(\cdot)) = \frac{m\|y_e - Cz(\cdot)\|_Y}{1 + \|y_e - Cz(\cdot)\|_Y} \quad .$$

Now observe that, since $Q(z) \in S \;\; \forall \, z \in X$, $Q$ is active under any projection $\mathcal{P}$ onto $S$ and, by (7.3.4), we have that

$$Q(z) = 0 \iff Cz(\cdot) \in \overline{B_\varepsilon(y_e)} \quad .$$

Thus, applying theorem 6.3.1, we get

$$z^* \in \nu(F) \iff \begin{cases} \xi(z^*) \in S \\ Cz^*(\cdot) \in \overline{B_\varepsilon(y_e)} \end{cases}$$

and therefore $F$ has necessary and sufficient conditions. Also observe that $F$ satisfies (c) since it includes the case of $\varepsilon$-estimated states.

Summarizing, $F: X \to X$ ,

$$F(z) = LNz(\cdot) + \mathcal{P}\xi(z) + q(z(\cdot))\overline{x} \tag{7.3.6}$$

satisfies all properties (a)-(d). Moreover,

$$(z_0, z(\cdot)) \mapsto q(z(\cdot))\overline{x}$$

is a completely continuous mapping and this may help when applying fixed point theorems. Clearly $F$ here is the analogous to the mapping $F$ in (6.4.8) for the problem of control.

7.3.2 - Properties (b)-(d) satisfied.

Let $q: \mathcal{F} \to \mathbb{R}$ be given by (7.3.5) with $m = 1$. Hence,

$$q(z(\cdot)) = 0 \iff Cz(\cdot) \in \overline{B_\varepsilon(y_e)}$$

$$0 \leq q(z(\cdot)) < 1 .$$

Now set $p: X \to \mathbb{R}$ the functional

$$p(z) = 1 - q(z(\cdot)) \quad , \quad z = (z_0, z(\cdot)).$$

Clearly $p$ satisfies

$$p(z) = 1 \iff Cz(\cdot) \in \overline{B_\varepsilon(y_e)} \tag{7.3.7}$$

and

$$0 < p(z) \leq 1 .$$

Now we can apply theorem 6.3.6 with

$$S' = S \; , \quad P = \mathcal{P} \quad \text{and} \quad \eta = \xi \; .$$

The mappings $F$ in (6.4.19) and (6.4.23) become

$$F(z) = LNz(\cdot) + p(x)\,\mathcal{P}\xi(x) \qquad (7.3.8)$$

$$F(z) = LNz(\cdot) + \mathcal{P}(p(x)\xi(x)) \;. \qquad (7.3.9)$$

Setting $f = I$ we have that $f(\nu(F)) = \nu(F)$. By (6.4.20) and (7.3.7),

$$z^* \in \nu(F) \;\; <= \begin{cases} \xi(z^*) \in S \\ Cz^*(\cdot) \in \overline{B_\varepsilon(y_e)} \end{cases}$$

which is the same as property (b), sufficiency of $F$. By (6.4.22) we have that

$$\left.\begin{aligned} z^* &\in \nu(F) \\ \xi(z^*) &\neq 0 \end{aligned}\right\} \;\; => \;\; \begin{cases} \xi(z^*) \in S \\ Cz^*(\cdot) \in \overline{B_\varepsilon(y_e)} \end{cases}$$

and this implies that F does not have necessary conditions. However,

$$E_S^\varepsilon = \{z^* \in \nu(F) : \xi(z^*) \neq 0\} \cup \{z^* \in \nu(F) : \xi(z^*) = 0 \text{ and } Cz^*(\cdot) \in \overline{B_\varepsilon(y_e)}\} \;.$$

So, we can obtain the $\varepsilon$-estimated states $E_S^\varepsilon$ once we have obtained the fixed points of $F$, $\nu(F)$. For the a posteriori test we can set (7.3.2), that is

$$Cz^*(\cdot) \in \overline{B_\varepsilon(y_e)} \qquad (7.3.10)$$

since every $z^* \in \nu(F)$ satisfies (7.3.1). Observe that the fixed points $z^* \in \nu(F)$ which will fail in the a posteriori test are the elements of the set

$$J_N(F) = \{z^* \in \nu(F) : \xi(z^*) = 0 \text{ and } Cz^*(\cdot) \notin \overline{B_\varepsilon(y_e)}\} .$$

Summarizing, both mappings F in (7.3.8) and (7.3.9) with a posteriori test (7.3.10) satisfy properties (b)-(d). These mappings are analogous to F in (6.4.29) and (6.4.30) for the problem of control.

7.3.3 - <u>The projection $P_e$</u> .

Further mappings F can be obtained if we consider the set

$$S_e = C^{-1}(\overline{B_\varepsilon(y_e)}) \subset \mathfrak{F} .$$

If $C:\mathfrak{F} \to Y$ is continuous then $S_e$ is closed and convex. Note that $C$ is continuous if and only if $C:Z \to Y$ (see section 3.1) is continuous. Also observe that (7.3.2) can be substituted by

$$z(\cdot) \in S_e \quad . \tag{7.3.11}$$

In the sequel we shall use $P_e:\mathfrak{F} \to \mathfrak{F}$

$$P_e = \text{any continuous projection onto } S_e$$

to obtain the analogous mappings to F in (6.4.10), (6.4.12) and (6.4.16).

7.3.4 - <u>All properties (a)-(d) satisfied</u>.

Here we consider the analogous mapping to $F$ in (6.4.10). Consider $F: X \to X$ given by

$$F(z) = (0, z(\cdot) - P_e z(\cdot)) + LNz(\cdot) + P\xi(z_0, P_e z(\cdot)) \tag{7.3.12}$$

and operative function $f: X \to X$ given by

$$f(z_0, z(\cdot)) = (z_0, P_e z(\cdot)) \quad .$$

Clearly $F$ satisfies properties (c) and (d). With a similar analysis to §6.4.3 we can verify that

$$f(\nu(F)) = E_s^\varepsilon$$

and therefore $F$ in (7.3.12) satisfies properties (a) and (b). In summary, $F$ satisfies all properties (a)-(d).

7.3.5 - <u>Properties (b)-(d) satisfied</u>.

Here we consider the analogous to $F$ in (6.4.12). Let $F$ be given by

$$F(z) = LNz(\cdot) + P\xi(z_0, P_e z(\cdot)) \tag{7.3.13}$$

and the operative set $f = I$ .

Now $F$ has a simpler form than in the previous paragraph but necessity does not hold anymore. The following a posteriori test must be satisfied by the fixed points of $F$ in order to be $\varepsilon$-estimated states,

$$Cz(\cdot) \in \overline{B_\varepsilon(y_e)}\ .$$

By doing a similar analysis to §6.4.4 we can verify that F satisfies properties (b)-(d). Moreover, the set $J_N(F)$ of the fixed points of F which will fail the a posteriori test is expected to be negligible comparing with the actual $\varepsilon$-estimated states $E_S^\varepsilon$ .

7.3.6 - <u>All properties satisfied using uniform projections.</u>

Suppose that $P':X \to X$ is a continuous projection onto S which satisfies

$P'$ is uniform in the first component.

We have seen in §4.7.2 that we can easily obtain such projections $P'$ by using characteristic functionals $\phi_S$ . As a matter of fact an example is given by $P' = \overline{P}'$ in (4.2.6) .

Now consider the following mapping $F:X \to X$ which is the analogous to F in (6.4.16) for the problem of control

$$F(z) = LNz(\cdot) + P'\xi(z_0, P_e z(\cdot)) \tag{7.3.14}$$

with operative function $f = I$ .

By doing a similar analysis to §6.4.5 we can verify that

$$\nu(F) = f(\nu(F)) = E_S^\varepsilon$$

and F satisfies all properties (a)-(d).

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